Meccanica
DOI 10.1007/s11012-013-9713-z

Residual stresses in random elastic composites: nonlocal micromechanics-based models and first estimates of the representative volume element size

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Received: 30 October 2012 / Accepted: 12 February 2013

Abstract Random elastic composites with residual stresses are examined in this paper with the aim of understanding how the prestress may influence the overall mechanical properties of the composite. A fully non-local effective response is found in perfect analogy with the un-prestressed case examined in (Drugan and Willis, J. Mech. Phys. Solids 44(4):497–524, 1996). The second gradient approximation is considered and the impact of the residual stresses on the estimate of the RVE size is studied whenever the local response is used to describe the mechanical properties of the heterogeneous medium. To this aim, total and incremental formulations are worked out in this paper and the influence of both uniform and spatially varying prestresses are studied. Among other results, it is shown how rapid oscillations of relatively "small" residual stresses in most cases may result in the impossibility of describing the overall behavior of the composite with a local constitutive equation. On the other hand, prestresses with relatively high amplitudes and slow spatial oscillations may even reduce the RVE size required for approximating the mechanical properties of un-prestressed heterogeneous media with a local constitutive equation.

Keywords Prestressed random composites · Residual stress · Micromechanics · Non-local elasticity · RVE size

1 Introduction

The necessity of detecting residual stresses in composites is often a crucial issue in order to be able to prevent undesired stress concentrations giving rise to local damage, debonding, pull out, etc. Although non destructive experimental techniques already applicable to detect residual stresses in metals, ceramics and other materials may be useful for resolving such stresses in composites, very little is known about their actual influence on the effective properties of the composite. The interplay between the microstructure of heterogeneous materials and the presence of residual stresses may be studied through a new field theory based on the distinction between macro and sub-macroscopic geometry through the approach developed in [8] and extended in [9, 10], although a more direct approach developed in [15] and extended in [13, 14, 20, 21] may be generalized to prestressed random elastic composites. This paper devotes attention to this problem by focusing on composites formed by randomly distributed 'small' inclusions in a matrix; the theory is developed

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Published online: 05 March 2013

for any shape although the examples are worked out for spherical inclusions or voids (see e.g. [25] for the influence of shape particles in the absence of residual stresses).

In the absence of residual stresses, micromechanics based effective explicit nonlocal constitutive equations have been obtained in [15] for the first time; there the effective response of random composites is evaluated starting from introducing a homogeneous comparison solid, i.e. a constant fourth order tensor of elastic moduli. Following [26], the stress generated by a compatible strain field superimposed on this medium is considered as a benchmark and the difference between the pointwise stress and such a benchmark is the *polarization stress*.

A probabilistic approach is then undertaken, where a sample space is considered and, for the generic realization of the composite a probability density is prescribed. For each phase forming the composite a characteristic function (also named indicator in the sequel) is introduced to indicate the presence of a point in a given phase for a given realization of the composite. This characteristic function is used (i) to approximate the moduli for the given realization through an ansatz where fourth order tensors of constant moduli for each phase multiply their corresponding indicator and (ii) to express the unknown polarization stresses in each phase.

One point and two points probability functions are then defined on the basis of the probability density cited above; the former generates a way to obtain the ensemble the average of such moduli and of the polarization, where the latter is carried over through the analysis and appears in the ensemble average of the Hashin and Strickman functional, governing the pointwise polarization stresses. Stationarity of such effective functional together with the averaging of the superimposed strain field obtained by solving the balance of linear momentum for the whole space lead to an integral equation for the polarization stresses in each phase in terms of the average of the superimposed strain field. The closed form solution of such system of equations in general is not possible and a space-Fourier transform approach is undertaken to successfully characterize solutions.

Indeed, in [15] several key results have been obtained, including a second strain gradient approximation of the integral response of the homogenized material, with particular reference to statistically and materially isotropic media. Among other results, the important issue of estimating the deviation from the second gradient effective response of the local term alone has been raised and specialized to the case of spherical inclusions. In particular, for a fixed deviation from nonlocality, both extensions and shears have been accounted for and estimates of the Representative Volume Element (RVE) size in both cases have been achieved. This procedure has been extended in the subsequent papers cited above, where also a fourth order approximation to the fully nonlocal response has been achieved. A recent paper [1] actually presents a very robust computational method for generating unit cells with randomly distributed inclusions consistent with the concept of RVE introduced in [15] and generalized in this paper in the presence of residual stresses.

Furthermore, the non-local effective model obtained in this paper permits improvement in the analysis of strain localization recently performed in [2–6] through introduction of size effect.

The key point of the method followed in the papers mentioned above is to introduce a comparison solid with constant elastic moduli which acts on the strain tensor, compatible with an underlying displacement field, and produces an idealized stress state. The difference between the pointwise stress and the latter is the *polarization stress* which in this case is unique.

For the sake of simplicity, two-phase composites are considered in the sequel although as in [15] the procedure is valid in general.

The presence of residual stresses introduces a source of non-uniqueness in the way in which the comparison solid may be defined. The analysis in the sequel shows that there are essentially four different families of options, here labeled with the letters $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{B}$, suitable to formulate and solve the problem of calculating the effective response of the composite.

To begin with, as in the classical approach for non prestressed composites, all such formulations are characterized by introducing a compatible strain field generating a reference stress through a fourth order tensor of constant elastic moduli. The compatibility of such strain is essential in order to express the stress-strain response in terms of the Green's function for the comparison medium defined on the whole three dimensional space (see e.g. [26], Appendix and [27], Sect. III.A.1, [15], Sect. 2).

An arbitrary reference state for the comparison solid may also be conceived: this could for instance coincide with ($\mathcal{A}_1$, Sect. 3.1) the actual residual stress, ($\mathcal{A}_2$, Sect. 3.2) zero or ($\mathcal{A}_3$, Sect. 3.3) neither of the above. A source of non uniqueness when choosing such a solid then seems to arise, due to the fact that the polarization stress needed to generate the whole stress field may have to comply with either choice. A Hashin and Strickman procedure analogous to that used in [15], namely when residual stresses are absent, is pursued for each case and the final averaged polarization stresses are obtained in Sect. 4. There it is proved that such stress turns out to be the same for each of the cases above and hence in this respect they are equivalent; such "family" of choices for the comparison solid is then labeled $\mathcal{A}$.

For two-phase composites, Sect. 4 shows that, in general, formulation $\mathcal{A}$ does not yield a fully averaged polarization stress, because the difference of the prestresses in the two phases appears explicitly.

Furthermore, an incremental formulation ($\mathcal{B}$, Sect. 4), based on an objective measure of the stress increment, is proposed. The resulting formulation loses the self-adjointness typical of all the other ones. As a result, a functional suitable to extract the approximations to the actual polarizations may formally be constructed in the same form delivered by the first variation of the Hashin and Strickman functional that could have been considered if major symmetries of the governing fourth order operator would occur.

Unlike formulation $\mathcal{A}$, the system of integral equations for $\mathcal{B}$ has the advantage that the governing polarizations in each phase depend upon fully averaged quantities, although a drawback of formulation $\mathcal{B}$ is that only homogeneously prestressed media may be treated with this approach.

2 Governing equations

It is known that the elastic behavior of a material when a strain $\mathbf{e}(\mathbf{x})$ is superimposed on a pre-existing stress state is defined by the following constitutive relation

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbb{L}(\mathbf{x})\mathbf{e}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x}), \tag{1}$$

where

- $\boldsymbol{\sigma}(\mathbf{x})$ is the Cauchy stress;
- $\mathbb{L}(\mathbf{x})$ is the fourth-order elastic tensor (exhibiting the minor and major symmetries, $\mathbb{L}_{ijkl} = \mathbb{L}_{jikl} = \mathbb{L}_{klij}$);
- $\mathbf{e}(\mathbf{x})$ is the symmetric part of $\mathbf{H}(\mathbf{x})$, the gradient of the superimposed displacement field $\mathbf{u}(\mathbf{x})$, i.e.

$$\mathbf{e}(\mathbf{x}) = \mathrm{sym}\big[\mathbf{H}(\mathbf{x})\big], \qquad \mathbf{H}(\mathbf{x}) = \nabla\mathbf{u} \tag{2}$$

- $\boldsymbol{\Sigma}(\mathbf{x})$ is a stress taking into account the effects given by the presence of a pre-existing stress [16],

$$\boldsymbol{\Sigma}(\mathbf{x}) = \mathbf{t}(\mathbf{x}) + \underbrace{\mathbf{H}(\mathbf{x})\mathbf{t}(\mathbf{x}) + \mathbf{t}(\mathbf{x})\mathbf{H}^T(\mathbf{x}) - \big[\mathrm{tr}\,\mathbf{H}(\mathbf{x})\big]\mathbf{t}(\mathbf{x})}_{\text{"geometrical terms"}} + \mathcal{D}\big[\mathbf{t}(\mathbf{x}), \mathbf{e}(\mathbf{x})\big], \tag{3}$$

where the superscript ^{T} denotes the transpose, $\mathbf{t}(\mathbf{x})$ represents the pre-existing (Cauchy) stress, and $\mathcal{D}$ is a sixth-order tensor such that

$$\mathbf{e}_1 \cdot \mathcal{D}\big[\mathbf{t}(\mathbf{x}), \mathbf{e}_2\big] = \mathbf{e}_2 \cdot \mathcal{D}\big[\mathbf{t}(\mathbf{x}), \mathbf{e}_1\big]$$

for any pair $\mathbf{e}_1, \mathbf{e}_2$ of second order symmetric tensors (see [16, 22–24]); in other words the fourth order tensor $\mathcal{D}[\mathbf{t}(\mathbf{x}), \cdot]$ resulting from the action of $\mathcal{D}$ on the prestress $\mathbf{t}(\mathbf{x})$, besides the obvious minor symmetries, it possesses the major ones too. Finally, the infinitesimal rigid rotation tensor $\mathbf{w}(\mathbf{x})$ is given by the decomposition rule

$$\mathbf{w}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) - \mathbf{e}(\mathbf{x}) = \mathrm{skw}\big[\mathbf{H}(\mathbf{x})\big]. \tag{4}$$

The linear constitutive relation (1) can be rewritten as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbb{C}^*(\mathbf{x})\mathbf{H}(\mathbf{x}) + \mathbf{t}(\mathbf{x}), \tag{5}$$

where[1]

$$\mathbb{C}^*(\mathbf{x}) = \mathbb{L}(\mathbf{x}) + \underbrace{\mathbf{I} \boxtimes \mathbf{t}(\mathbf{x}) + \big(\mathbf{t}(\mathbf{x}) \boxtimes \mathbf{I}\big)\mathbb{T}^T - \mathbf{t}(\mathbf{x}) \otimes \mathbf{I}}_{\text{"geometrical terms"}} + \mathcal{D}\big[\mathbf{t}(\mathbf{x}), \cdot\big], \tag{6}$$

where $\mathbb{T}^T = \sum_{ijkl} \delta_{jk}\delta_{il}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$ is the "transposer" operator, namely the fourth order tensor allowing for transposing any second order tensor. It is worth noting the tensor $\mathbb{C}^*(\mathbf{x})$ then exhibits left minor symmetries only, namely $\mathbb{C}^*_{ijkl} = \mathbb{C}^*_{jikl} \neq \mathbb{C}^*_{ijlk}$.

Equation (6) demonstrates that the response can be anisotropic due to the presence of non-zero prestress $\mathbf{t}(\mathbf{x})$ even in the case of initially isotropic materials.

Balance equations may be written for the stresses and the body forces in two possible ways.

[1] Here $(\mathbf{A} \boxtimes \mathbf{B})\mathbf{U} := \mathbf{A}\mathbf{U}\mathbf{B}^T$, for any triple of second order tensors $\mathbf{A}, \mathbf{B}, \mathbf{U}$.

- *Total formulation.* Considering the presence of a body force vector $\mathbf{f}(\mathbf{x})$, the equilibrium of the medium is expressed by

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{7}$$

from which the equilibrium condition for the body when no superimposed displacement field is considered, $\mathbf{u} = \mathbf{0}$, follows

$$\operatorname{div}\mathbf{t}(\mathbf{x}) + \mathbf{f}^*(\mathbf{x}) = \mathbf{0}, \tag{8}$$

where $\mathbf{f}^*(\mathbf{x})$ represents the pre-existing body force.
- *Incremental formulation.* An objective measure of increment in the Cauchy stress may be introduced by looking at the structure of the constitutive equation (5) and (3):

$$\overset{\circ}{\boldsymbol{\sigma}}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x}) - \mathbf{t}(\mathbf{x}) - \mathbf{w}(\mathbf{x})\mathbf{t}(\mathbf{x}) + \mathbf{t}(\mathbf{x})\mathbf{w}(\mathbf{x}), \tag{9}$$

which is insensitive to the rigid body rotations. For this stress increment a linear constitutive relation may be provided by replacing $\boldsymbol{\sigma}$ with (5) to get the following expression:

$$\overset{\circ}{\boldsymbol{\sigma}}(\mathbf{x}) = \mathbb{C}(\mathbf{x})\mathbf{e}(\mathbf{x}), \tag{10}$$

where

$$\mathbb{C}(\mathbf{x}) = \mathbb{L}(\mathbf{x}) + \underbrace{\mathbf{I}\boxtimes\mathbf{t}(\mathbf{x}) + \mathbf{t}(\mathbf{x})\boxtimes\mathbf{I} - \mathbf{t}(\mathbf{x})\otimes\mathbf{I}}_{\text{“geometrical terms”}} + \mathcal{D}\big[\mathbf{t}(\mathbf{x}),\cdot\big]. \tag{11}$$

Unlike $\mathbb{C}^*(\mathbf{x})$ introduced above, the tensor $\mathbb{C}(\mathbf{x})$ exhibits both minor symmetries, namely $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$, although major symmetries are still not present;[2] from now on this entity will be carried over to represent the elastic constitutive information, acting on the pure strain variable $\mathbf{e}(\mathbf{x})$, accounting for the presence of prestress.

Considering relations (7)–(8), the equilibrium condition can be obtained in terms of the stress increment $\overset{\circ}{\boldsymbol{\sigma}}(\mathbf{x})$,

$$\operatorname{div}\overset{\circ}{\boldsymbol{\sigma}}(\mathbf{x}) + \overset{\circ}{\mathbf{f}}(\mathbf{x}) = \mathbf{0}, \tag{12}$$

where an objective measure of the increment in the body force, $\overset{\circ}{\mathbf{f}}$, has been introduced

$$\overset{\circ}{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{f}^*(\mathbf{x}) + \operatorname{div}\big[\mathbf{w}(\mathbf{x})\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x})\mathbf{w}(\mathbf{x})\big]. \tag{13}$$

3 ‘Total’ formulations

The governing equations introduced in terms of the total quantities may allow for evaluating the effective response for heterogeneous materials by following procedures analogous to the one introduced by Drugan and Willis in [15]. There, a comparison solid with constant moduli was introduced and considered to fill the entire three dimensional space $\mathbb{R}^3$ and a Hashin and Shtrikman functional was singled out to find the effective response for an elastic random composite.

Then, the effective response is obtained by considering the entire space to be filled by a comparison medium with (homogeneous) constitutive tensor $\mathbb{L}_0$, on which loading is performed through the body force $\mathbf{f}(\mathbf{x})$ and for which the solution of the field equations (7), (1), with $\boldsymbol{\Sigma}(\mathbf{x}) \equiv \mathbf{0}$, is given by $\{\boldsymbol{\sigma}_0(\mathbf{x}), \mathbf{e}_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x})\}$, where $\mathbf{e}_0(\mathbf{x}) := \operatorname{sym}[\nabla\mathbf{u}_0(\mathbf{x})]$.

In the presence of residual stresses obviously $\boldsymbol{\Sigma}(\mathbf{x}) \neq \mathbf{0}$ and phases are described by the linearized constitutive relation (1); the procedure highlighted above may also apply, provided that the prestress will be carried through the entire analysis. The introduced multiple choices for the comparison medium, giving rise to apparently different formulations, which will be denoted by $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. In spite of this possible non-uniqueness, in the sequel it is shown that all three approaches turn out to be equivalent.

3.1 Formulation $\mathcal{A}_1$

The linearized constitutive relation (1) can be rewritten as follows:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbb{L}_0\mathbf{e}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x}), \tag{14}$$

where $\boldsymbol{\tau}(\mathbf{x})$ is the stress polarization field,

$$\boldsymbol{\tau}(\mathbf{x}) = \big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]\mathbf{e}(\mathbf{x}). \tag{15}$$

This choice entails having the entire residual stress actually present in the real material to prestress the chosen comparison solid.

[2] We note that $\mathbb{C}^*(\mathbf{x})\mathbf{e}(\mathbf{x}) = \mathbb{C}(\mathbf{x})\mathbf{e}(\mathbf{x})$, since $\mathbf{e}(\mathbf{x}) = \operatorname{sym}[\mathbf{H}(\mathbf{x})]$.

By substituting the linearized constitutive equation (14) into the balance of linear momentum (7) we obtain the following expression:

$$\mathrm{div}\big[\mathbb{L}_0\mathbf{e}(\mathbf{x})\big] + \mathbf{f}(\mathbf{x}) + \mathrm{div}\big[\boldsymbol{\tau}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x})\big] = 0, \tag{16}$$

whose solution for the superimposed deformation field $\mathbf{e}(\mathbf{x})$ is given as follows:

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big[\boldsymbol{\tau}(\mathbf{x}') + \boldsymbol{\Sigma}(\mathbf{x}')\big]\,\mathrm{d}\mathbf{x}', \tag{17}$$

after adapting [28], where

$$\big[\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big]_{ijkl} = \left.\frac{\partial^2[\mathbf{G}_0(\mathbf{x}-\mathbf{x}')]_{jk}}{\partial x_i \partial x'_l}\right|_{(ij),(kl)}. \tag{18}$$

In Eq. (18), (ij) and (kl) stand for symmetrization on these indexes and $\mathbf{G}_0(\mathbf{x})$ is the infinite—homogeneous—body Green's function for the comparison material given as solution of

$$\mathbb{L}_{0\,ijkl}\frac{\partial^2[\mathbf{G}_0(\mathbf{x})]_{jm}}{\partial x_i \partial x_l} + \delta_{km}\delta(\mathbf{x}) = 0, \tag{19}$$

where δ_{km} is the Kronecker delta and $\delta(\mathbf{x})$ is the three-dimensional Dirac delta function. Equation (17) exhibits a fully nonlocal character, which is encountered in [15] for non-prestressed random elastic composites and in [7] for thin films.

Using the definition (15) of the stress polarization in Eq. (17), we obtain

$$\mathbf{e}_0(\mathbf{x}) = \big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\boldsymbol{\tau}(\mathbf{x}) + \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big[\boldsymbol{\tau}(\mathbf{x}') + \boldsymbol{\Sigma}(\mathbf{x}')\big]\mathrm{d}\mathbf{x}', \tag{20}$$

in which the stress polarization $\boldsymbol{\tau}(\mathbf{x})$ is the stationary value for the functional written below:

$$\mathcal{H}\big[\boldsymbol{\tau}(\mathbf{x})\big] = \int_{\mathbb{R}^3}\Big\{\boldsymbol{\tau}(\mathbf{x})\cdot\big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\boldsymbol{\tau}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x})\cdot\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\tau}(\mathbf{x}')\mathrm{d}\mathbf{x}' - 2\boldsymbol{\tau}(\mathbf{x})\cdot\Big[\mathbf{e}_0(\mathbf{x}) - \int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\Sigma}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\Big]\Big\}\,\mathrm{d}\mathbf{x}. \tag{21}$$

This formulation is used to characterize the effective response of random composites as in [15]. Related issues arising in systems with uncertain parameters are treated in [11, 12]. Considering a sample space $\mathcal{S}$ in which α represents the individual member, the characteristic function $\chi_r(\mathbf{x};\alpha)$ defines the presence in the point $\mathbf{x}$ of phase r for the realization α,

$$\chi_r(\mathbf{x};\alpha) = \begin{cases} 1, & \text{if } \mathbf{x} \in \text{phase } r;\\ 0, & \text{if } \mathbf{x} \notin \text{phase } r. \end{cases} \tag{22}$$

Since the ensemble average $\langle f(\mathbf{x})\rangle$ of a function $f(\mathbf{x})$ is defined as

$$\big\langle f(\mathbf{x})\big\rangle \equiv \int_{\mathcal{S}} f(\mathbf{x};\alpha)\,p(\alpha)\,\mathrm{d}\alpha, \tag{23}$$

where $p(\alpha)$ represents the probability density of the realization α within the sample space $\mathcal{S}$, it follows that the (one-point) probability to have phase r at $\mathbf{x}$ is

$$P_r(\mathbf{x}) = \big\langle\chi_r(\mathbf{x})\big\rangle \equiv \int_{\mathcal{S}}\chi_r(\mathbf{x};\alpha)\,p(\alpha)\,\mathrm{d}\alpha, \tag{24}$$

while the (two-point) probability to have phase r at $\mathbf{x}$ and simultaneously phase s at $\mathbf{x}'$ is

$$\begin{aligned} P_{rs}(\mathbf{x},\mathbf{x}') &= \big\langle\chi_r(\mathbf{x})\chi_s(\mathbf{x}')\big\rangle \\ &\equiv \int_{\mathcal{S}}\chi_r(\mathbf{x};\alpha)\chi_s(\mathbf{x}';\alpha)\,p(\alpha)\,\mathrm{d}\alpha \\ &= P_{rs}(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{25}$$

Restricting attention to composite materials with homogeneous phases (i.e. each phase r is characterized by constant $\mathbb{L}_r$ with $r = 1,\ldots,n$), the fourth-order elastic tensor $\mathbb{L}(\mathbf{x},\alpha)$ and its ensemble average are

$$\mathbb{L}(\mathbf{x};\alpha) = \sum_{r=1}^{n}\mathbb{L}_r\,\chi_r(\mathbf{x};\alpha) \quad\Rightarrow\quad \big\langle\mathbb{L}(\mathbf{x})\big\rangle = \sum_{r=1}^{n}\mathbb{L}_r\,P_r(\mathbf{x}). \tag{26}$$

The polarization stress field $\boldsymbol{\tau}$ is chosen to have the following form:

$$\boldsymbol{\tau}(\mathbf{x};\alpha) = \sum_{r=1}^{n}\boldsymbol{\tau}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha), \tag{27}$$

which, as discussed in [15], is the most general representation for one-point and two-points probability correlations in (21). This will provide an approximation for the stress polarization, in which the functions $\boldsymbol{\tau}_r(\mathbf{x})$ will be determined in the sequel through a variational argument based on the probabilistic version of (21).

In order to evaluate the ensemble average of the probabilistic version of (21), unlike for the case of non prestressed composites, the ansatz must be considered also for the following fields

– the pre-existing Cauchy stress $\mathbf{t}$,
– the gradient of superimposed displacement $\mathbf{H}$;

from which the ansatz for the sixth-order tensor $\mathcal{D}$ follows. Henceforth, we have

$$\mathbf{t}(\mathbf{x};\alpha)=\sum_{r=1}^{n}\mathbf{t}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha),\qquad \mathbf{H}(\mathbf{x};\alpha)=\sum_{r=1}^{n}\mathbf{H}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha),\tag{28}$$

from which the symmetric and skew symmetric part of $\mathbf{H}(\mathbf{x};\alpha)$ follow:

$$\mathbf{e}(\mathbf{x};\alpha)=\sum_{r=1}^{n}\mathbf{e}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha),\qquad \mathbf{w}(\mathbf{x};\alpha)=\sum_{r=1}^{n}\mathbf{w}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha),\tag{29}$$

and hence the stress $\boldsymbol{\Sigma}(\mathbf{x})$, Eq. (3), for the realization α follows

$$\boldsymbol{\Sigma}(\mathbf{x};\alpha)=\sum_{r=1}^{n}\boldsymbol{\Sigma}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha).\tag{30}$$

Restricting attention to statistically uniform composites, the one-point $P_r(\mathbf{x})$, Eq. (24), and two-point $P_{rs}(\mathbf{x},\mathbf{x}')$, Eq. (25), probabilities are not affected by translations and making the ergodic assumption we have

$$P_r(\mathbf{x})=P_r=c_r,\qquad P_{rs}(\mathbf{x},\mathbf{x}')=P_{rs}(\mathbf{x}-\mathbf{x}'),\tag{31}$$

where c_r is the volume concentration of the phase r. Results on the autocorrelation function for polycrystals may be found in [18], whereas for the case of non overlapping spheres they may be found in [19].

Using Eqs. (28), (29), (30), (31) in the ensemble average of the functional (21) yields

$$\begin{aligned}\langle\mathcal{H}[\boldsymbol{\tau}(\mathbf{x})]\rangle &=\sum_{r=1}^{n}c_r\int_{\mathbb{R}^3}\boldsymbol{\tau}_r(\mathbf{x})\cdot\left\{\delta\mathbb{L}_r^{-1}\boldsymbol{\tau}_r(\mathbf{x})-2\mathbf{e}_0(\mathbf{x})\right\}\mathrm{d}\mathbf{x}\\ &+\sum_{r,s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\tau}_r(\mathbf{x})\cdot\left\{\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\right.\\ &\left.\times\left[\boldsymbol{\tau}_s(\mathbf{x}')+2\boldsymbol{\Sigma}_s(\mathbf{x}')\right]P_{rs}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right\}\mathrm{d}\mathbf{x},\end{aligned}\tag{32}$$

which is stationary when $(r=1,\ldots,n)$

$$\begin{aligned}c_r\mathbf{e}_0(\mathbf{x})&=c_r\delta\mathbb{L}_r^{-1}\boldsymbol{\tau}_r(\mathbf{x})\\ &+\sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\left[\boldsymbol{\tau}_s(\mathbf{x}')+\boldsymbol{\Sigma}_s(\mathbf{x}')\right]\\ &\times P_{rs}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}',\end{aligned}\tag{33}$$

where we set:

$$\delta\mathbb{L}_r=\mathbb{L}_r-\mathbb{L}_0.\tag{34}$$

Equations (28)–(29) may be substituted in the expression (17) in order to obtain $\mathbf{e}_0$, so that the ensemble average of such field leads to the following expression relating $\mathbf{e}_0$, the average strain $\langle\mathbf{e}\rangle$ and the (unknown) polarizations of each phase:

$$\begin{aligned}\langle\mathbf{e}\rangle(\mathbf{x})&=\mathbf{e}_0(\mathbf{x})-\sum_{s=1}^{n}c_s\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\\ &\times\left[\boldsymbol{\tau}_s(\mathbf{x}')+\boldsymbol{\Sigma}_s(\mathbf{x}')\right]\mathrm{d}\mathbf{x}',\end{aligned}\tag{35}$$

which is used together with the stationarity condition (33) to get the following system of integral equations for $\boldsymbol{\tau}_r(\mathbf{x})$ $(r=1,\ldots,n)$ in terms of $\mathbf{e}_s(\mathbf{x})$ and $\boldsymbol{\Sigma}_s(\mathbf{x})$,

$$\begin{aligned}c_r\langle\mathbf{e}\rangle(\mathbf{x})&=c_r\delta\mathbb{L}_r^{-1}\boldsymbol{\tau}_r(\mathbf{x})\\ &+\sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\left[\boldsymbol{\tau}_s(\mathbf{x}')+\boldsymbol{\Sigma}_s(\mathbf{x}')\right]\\ &\times\left[P_{rs}(\mathbf{x}-\mathbf{x}')-c_rc_s\right]\mathrm{d}\mathbf{x}'.\end{aligned}\tag{36}$$

Once the system of integral equations (36) is solved for the unknowns $\boldsymbol{\tau}_r(\mathbf{x})$ $(r=1,\ldots,n)$, the ensemble averaged polarization can be obtained

$$\langle\boldsymbol{\tau}\rangle(\mathbf{x})=\sum_{r=1}^{n}c_r\boldsymbol{\tau}_r(\mathbf{x}),\tag{37}$$

and it can be used in the ensemble average of the constitutive equation (14),

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x})=\mathbb{L}_0\langle\mathbf{e}\rangle(\mathbf{x})+\langle\boldsymbol{\tau}\rangle(\mathbf{x})+\langle\boldsymbol{\Sigma}\rangle(\mathbf{x}).\tag{38}$$

3.2 Formulation $\mathcal{A}_2$

The constitutive relation (1) may be recast in such a way a new choice of the polarization stress may contain the entire residual stress, so that the comparison medium remains completely un-prestressed. In this case such equation may be written in the usual form

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbb{L}_0\mathbf{e}(\mathbf{x}) + \mathbf{p}(\mathbf{x}), \tag{39}$$

where the polarization field $\mathbf{p}(\mathbf{x})$ has been introduced such that

$$\mathbf{p}(\mathbf{x}) = \left[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\right]\mathbf{e}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x}) \tag{40}$$

holds.

The fact that this polarization contains the prestress changes the effective response in a nontrivial way, as will be shown in the sequel. Furthermore, although the choice just made is different from (14) made in the previous section for $\mathcal{A}_1$, the methodology carried through this and the following section remains the same as the one introduced before, actually based on [15].

Henceforth, by substituting the constitutive equation (39) in the balance of linear momentum (7) we obtain the following expression:

$$\operatorname{div}\left[\mathbb{L}_0\mathbf{e}(\mathbf{x})\right] + \mathbf{f}(\mathbf{x}) + \operatorname{div}\mathbf{p}(\mathbf{x}) = 0, \tag{41}$$

whose solution for the superimposed deformation field $\mathbf{e}(\mathbf{x})$ is given as follows

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}\left(\mathbf{x}'\right)\mathrm{d}\mathbf{x}', \tag{42}$$

again by adapting the argument given in the previous section for Eq. (17).

Using the definition (40) of the stress polarization in Eq. (42), we obtain

$$\mathbf{e}_0(\mathbf{x}) = \left[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\right]^{-1}\left[\mathbf{p}(\mathbf{x}) - \boldsymbol{\Sigma}(\mathbf{x})\right] + \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}\left(\mathbf{x}'\right)\mathrm{d}\mathbf{x}', \tag{43}$$

in which $\mathbf{p}(\mathbf{x})$ is the stationary value for the functional

$$\mathcal{H}\left[\mathbf{p}(\mathbf{x})\right] = \int_{\mathbb{R}^3}\left\{\mathbf{p}(\mathbf{x})\cdot\left[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\right]^{-1}\mathbf{p}(\mathbf{x}) + \mathbf{p}(\mathbf{x})\cdot\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}\left(\mathbf{x}'\right)\mathrm{d}\mathbf{x}' - 2\mathbf{p}(\mathbf{x})\cdot\left[\mathbf{e}_0(\mathbf{x}) + \left[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\right]^{-1}\boldsymbol{\Sigma}(\mathbf{x})\right]\right\}\mathrm{d}\mathbf{x}. \tag{44}$$

Using Eqs. (27), (31) and an ansatz for the polarization $\mathbf{p}$, Eq. (40), analogous to that used in formulation $\mathcal{A}_1$, Eq. (28), i.e.

$$\mathbf{p}(\mathbf{x};\alpha) = \sum_{r=1}^{n}\mathbf{p}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha), \tag{45}$$

the ensemble average of the functional (44) may be expressed in the following form by

$$\begin{aligned}\langle\mathcal{H}\left[\mathbf{p}(\mathbf{x})\right]\rangle &= \sum_{r=1}^{n} c_r \int_{\mathbb{R}^3}\mathbf{p}_r(\mathbf{x}) \times\left\{\delta\mathbb{L}_r^{-1}\left[\mathbf{p}_r(\mathbf{x}) - 2\boldsymbol{\Sigma}_r(\mathbf{x})\right] - 2\mathbf{e}_0(\mathbf{x})\right\}\mathrm{d}\mathbf{x} \\ &\quad + \sum_{r,s=1}^{n}\int_{\mathbb{R}^3}\mathbf{p}_r(\mathbf{x}) \times\left\{\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}_s\left(\mathbf{x}'\right)P_{rs}\left(\mathbf{x}-\mathbf{x}'\right)\mathrm{d}\mathbf{x}'\right\}\mathrm{d}\mathbf{x};\end{aligned} \tag{46}$$

this turns out to be stationary whenever the following relation holds ($r = 1,\ldots,n$)

$$c_r\mathbf{e}_0(\mathbf{x}) = c_r\delta\mathbb{L}_r^{-1}\left[\mathbf{p}_r(\mathbf{x}) - \boldsymbol{\Sigma}_r(\mathbf{x})\right] + \sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}_s\left(\mathbf{x}'\right)P_{rs}\left(\mathbf{x}-\mathbf{x}'\right)\mathrm{d}\mathbf{x}'. \tag{47}$$

By using Eqs. (28), (29) in the ensemble averaging of the superimposed deformation field (42) the following relation between $\mathbf{e}_0$, the averaged strain and the (unknown) polarizations of each phase results:

$$\langle\mathbf{e}\rangle(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \sum_{s=1}^{n} c_s\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0\left(\mathbf{x}-\mathbf{x}'\right)\mathbf{p}_s\left(\mathbf{x}'\right)\mathrm{d}\mathbf{x}'. \tag{48}$$

The latter equation together with the stationarity condition (33) yield the following system of integral equations for $\mathbf{p}_r(\mathbf{x})$ ($r = 1,\ldots,n$) in terms of $\mathbf{e}_s(\mathbf{x})$, and $\boldsymbol{\Sigma}_s(\mathbf{x})$,

$$c_r\langle\mathbf{e}\rangle(\mathbf{x}) = c_r\delta\mathbb{L}_r^{-1}\big[\mathbf{p}_r(\mathbf{x}) - \boldsymbol{\Sigma}_r(\mathbf{x})\big] + \sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\mathbf{p}_s(\mathbf{x}') \times\big[P_{rs}(\mathbf{x}-\mathbf{x}') - c_r c_s\big]\,\mathrm{d}\mathbf{x}'. \tag{49}$$

Once the system of integral equations (36) is solved for the unknowns $\mathbf{p}_r(\mathbf{x})$ $(r = 1, \ldots, n)$, the ensemble average can be obtained

$$\langle\mathbf{p}\rangle(\mathbf{x}) = \sum_{r=1}^{n} c_r\mathbf{p}_r(\mathbf{x}), \tag{50}$$

and it can be used in the ensemble average of the constitutive equation (39),

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \mathbb{L}_0\langle\mathbf{e}\rangle(\mathbf{x}) + \langle\mathbf{p}\rangle(\mathbf{x}). \tag{51}$$

3.3 Formulation $\mathcal{A}_3$

As a third possibility, the comparison solid may be chosen to be in an arbitrarily prestressed state which then may not necessarily coincide with the corresponding field existing in the actual material. Although this may seem a bit artificial, the related averaged response of the composite obtained in this way will show that this approach is actually equivalent to the previous two.

In this case, the constitutive relation (1) may be rewritten as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbb{L}_0\mathbf{e}(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) + \mathbf{q}(\mathbf{x}), \tag{52}$$

where the stress polarization field $\mathbf{q}(\mathbf{x})$ and a comparison pre-existing stress $\boldsymbol{\Sigma}_0$ have been introduced,

$$\mathbf{q}(\mathbf{x}) = \big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]\mathbf{e}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x}) - \boldsymbol{\Sigma}_0(\mathbf{x}). \tag{53}$$

In this case the polarization contains the difference of the prestress in the two phases.

Using the linearized constitutive equation (52) in the equilibrium equation (7) yields

$$\mathrm{div}\big[\mathbb{L}_0\mathbf{e}(\mathbf{x})\big] + \mathbf{f}(\mathbf{x}) + \mathrm{div}\,\boldsymbol{\Sigma}_0(\mathbf{x}) + \mathrm{div}\,\mathbf{q}(\mathbf{x}) = 0, \tag{54}$$

whose solution for the superimposed deformation field $\mathbf{e}(\mathbf{x})$ is given as follows

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big[\boldsymbol{\Sigma}_0(\mathbf{x}') + \mathbf{q}(\mathbf{x}')\big]\,\mathrm{d}\mathbf{x}'. \tag{55}$$

Using the definition (53) of the stress polarization in Eq. (55), we obtain

$$\mathbf{e}_0(\mathbf{x}) = \big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\big[\mathbf{q}(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) - \boldsymbol{\Sigma}(\mathbf{x})\big] + \int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big[\boldsymbol{\Sigma}_0(\mathbf{x}') + \mathbf{q}(\mathbf{x}')\big]\,\mathrm{d}\mathbf{x}', \tag{56}$$

in which the stress polarization $\mathbf{q}(\mathbf{x})$ is the stationary value for the functional

$$\begin{aligned}\mathcal{H}\big[\mathbf{q}(\mathbf{x})\big] &= \int_{\mathbb{R}^3}\Big\{\mathbf{q}(\mathbf{x})\cdot\big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\mathbf{q}(\mathbf{x}) \\ &+ \mathbf{q}(\mathbf{x})\cdot\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\big[2\boldsymbol{\Sigma}_0(\mathbf{x}') + \mathbf{q}(\mathbf{x}')\big]\,\mathrm{d}\mathbf{x}' \\ &- 2\mathbf{q}(\mathbf{x})\cdot\big[\mathbf{e}_0(\mathbf{x}) + \big[\mathbb{L}(\mathbf{x}) - \mathbb{L}_0\big]^{-1} \\ &\times\big[\boldsymbol{\Sigma}(\mathbf{x}) - \boldsymbol{\Sigma}_0(\mathbf{x})\big]\big]\Big\}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{57}$$

Using Eqs. (28), (31) and the following ansatz for the polarization stress field,

$$\mathbf{q}(\mathbf{x};\alpha) = \sum_{r=1}^{n}\mathbf{q}_r(\mathbf{x})\chi_r(\mathbf{x};\alpha), \tag{58}$$

the ensemble average of the functional (57) is expressed by

$$\begin{aligned}\big\langle\mathcal{H}\big[\mathbf{q}(\mathbf{x})\big]\big\rangle &= \sum_{r=1}^{n} c_r\int_{\mathbb{R}^3}\mathbf{q}_r(\mathbf{x})\cdot\Big\{\delta\mathbb{L}_r^{-1}\big[\mathbf{q}_r(\mathbf{x}) + 2\boldsymbol{\Sigma}_0(\mathbf{x}) \\ &- 2\boldsymbol{\Sigma}_r(\mathbf{x})\big] - 2\mathbf{e}_0(\mathbf{x}) \\ &+ 2\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\Sigma}_0(\mathbf{x}')\mathrm{d}\mathbf{x}'\Big\}\,\mathrm{d}\mathbf{x} \\ &+ \sum_{r,s=1}^{n}\int_{\mathbb{R}^3}\mathbf{q}_r(\mathbf{x}) \\ &\times\Big\{\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\mathbf{q}_s(\mathbf{x}')P_{rs}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}'\Big\}\,\mathrm{d}\mathbf{x},\end{aligned} \tag{59}$$

which is stationary when $(r = 1, \ldots, n)$

$$c_r\mathbf{e}_0(\mathbf{x}) = c_r\Big\{\delta\mathbb{L}_r^{-1}\big[\mathbf{q}_r(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) - \boldsymbol{\Sigma}_r(\mathbf{x})\big]$$

$$+\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\Sigma}_0(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\bigg\}$$
$$+\sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\mathbf{q}_s(\mathbf{x}')P_{rs}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{60}$$

Making use of Eqs. (28), (29) in the ensemble averaging of the superimposed deformation field (55), a relationship among $\mathbf{e}_0$, the average strain field $\langle\mathbf{e}\rangle(\mathbf{x})$ and the (unknown) polarizations $\mathbf{q}_r(\mathbf{x})$, $(r = 1,\ldots,n)$, is obtained:

$$\langle\mathbf{e}\rangle(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \sum_{s=1}^{n}c_s\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\mathbf{q}_s(\mathbf{x}')\,\mathrm{d}\mathbf{x}'$$
$$-\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\Sigma}_0(\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{61}$$

This relation is used together with the stationarity condition (60) to get the following system of integral equations for $\mathbf{q}_r(\mathbf{x})$ $(r = 1,\ldots,n)$ in terms of $\mathbf{e}_s(\mathbf{x})$ and $\boldsymbol{\Sigma}_s(\mathbf{x})$,

$$c_r\langle\mathbf{e}\rangle(\mathbf{x}) = c_r\delta\mathbb{L}_r^{-1}\big[\mathbf{q}_r(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) - \boldsymbol{\Sigma}_r(\mathbf{x})\big]$$
$$+\sum_{s=1}^{n}\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\mathbf{q}_s(\mathbf{x}')$$
$$\times\big[P_{rs}(\mathbf{x}-\mathbf{x}') - c_rc_s\big]\,\mathrm{d}\mathbf{x}'. \tag{62}$$

Once the system of integral equations (62) is solved for the unknowns $\mathbf{q}_r(\mathbf{x})$ $(r = 1,\ldots,n)$, the ensemble average can be obtained

$$\langle\mathbf{q}\rangle(\mathbf{x}) = \sum_{r=1}^{n}c_r\mathbf{q}_r(\mathbf{x}), \tag{63}$$

and it can be used in the ensemble average of the linearized constitutive equation (52),

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \mathbb{L}_0\langle\mathbf{e}\rangle(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) + \langle\mathbf{q}\rangle(\mathbf{x}). \tag{64}$$

4 Incremental approach: formulation $\mathcal{B}$

A comparison solid may be introduced for the composite by looking at the incremental formulation stated in Sect. 2. In particular, relation (10) may be rewritten as follows

$$\overset{\circ}{\boldsymbol{\sigma}}(\mathbf{x}) = \mathbb{L}_0\mathbf{e}(\mathbf{x}) + \boldsymbol{\rho}(\mathbf{x}), \tag{65}$$

where the stress polarization field $\boldsymbol{\rho}(\mathbf{x})$ has been introduced.

Although $\mathbb{C}(\mathbf{x})$, Eq. (11), has minor symmetries only, we recall that the comparison medium is here defined by a constant tensor $\mathbb{L}_0$ exhibiting major and both minor symmetries.

From comparison of Eq. (65) with Eq. (10), the polarization field follows

$$\boldsymbol{\rho}(\mathbf{x}) = \big[\mathbb{C}(\mathbf{x}) - \mathbb{L}_0\big]\mathbf{e}(\mathbf{x}). \tag{66}$$

Using the incremental linearized constitutive relation (65) in the balance of linear momentum (7) we obtain the following equation:

$$\operatorname{div}\big[\mathbb{L}_0\mathbf{e}(\mathbf{x})\big] + \overset{\circ}{\mathbf{f}}(\mathbf{x}) + \operatorname{div}\boldsymbol{\rho}(\mathbf{x}) = 0. \tag{67}$$

By adapting the argument given in Sect. 2 for (17) we have that the solution of (67) for the superimposed deformation field $\mathbf{e}(\mathbf{x})$ may be expressed as follows:

$$\mathbf{e}(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\rho}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{68}$$

Using the definition (66) of the stress polarization in Eq. (68), we obtain

$$\mathbf{e}_0(\mathbf{x}) = \big[\mathbb{C}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\boldsymbol{\rho}(\mathbf{x})$$
$$+\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\rho}(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{69}$$

which, unlike Eqs. (20), (43), (56), is characterized by the non-self adjoint operator $[\mathbb{C}(\mathbf{x}) - \mathbb{L}_0]^{-1}$. For this reason a Hashin and Shtrikman functional for which (69) represents its stationary point cannot be constructed. Nevertheless, a weak form of such integral equation may be considered by introducing a field of 'virtual' polarizations $\boldsymbol{\rho}^*(\mathbf{x})$ and by defining a new functional $\mathcal{K}[\boldsymbol{\rho}(\mathbf{x});\boldsymbol{\rho}^*(\mathbf{x})]$ in the following form:

$$\mathcal{K}\big[\boldsymbol{\rho}(\mathbf{x});\boldsymbol{\rho}^*(\mathbf{x})\big]$$
$$=\int_{\mathbb{R}^3}\bigg\{\boldsymbol{\rho}^*(\mathbf{x})\cdot\big[\mathbb{C}(\mathbf{x}) - \mathbb{L}_0\big]^{-1}\boldsymbol{\rho}(\mathbf{x}) + \boldsymbol{\rho}^*(\mathbf{x})$$
$$\times\int_{\mathbb{R}^3}\boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}')\boldsymbol{\rho}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' - \boldsymbol{\rho}^*(\mathbf{x})\cdot\mathbf{e}_0(\mathbf{x})\bigg\}\,\mathrm{d}\mathbf{x}; \tag{70}$$

whenever such functional achieves value equal to zero for arbitrary choices of the test field $\boldsymbol{\rho}^*(\mathbf{x})$ Eq. (69) is in fact recovered.

Restricting attention to composite materials with homogeneous phases uniformly prestressed[3] (i.e. each phase r is characterized by constant $\mathbb{C}_r$ with $r = 1, \dots, n$), the fourth-order elastic tensor $\mathbb{C}(\mathbf{x}, \alpha)$ and its ensemble average are

$$\mathbb{C}(\mathbf{x};\alpha) = \sum_{r=1}^{n} \mathbb{C}_r \chi_r(\mathbf{x};\alpha)$$
$$\Rightarrow \quad \langle \mathbb{C}(\mathbf{x}) \rangle = \sum_{r=1}^{n} \mathbb{C}_r P_r(\mathbf{x}), \tag{71}$$

and using Eqs. (28), (31) and the following ansatz for the polarization stress and its test counterpart,

$$\begin{aligned} \boldsymbol{\rho}(\mathbf{x};\alpha) &= \sum_{r=1}^{n} \boldsymbol{\rho}_r(\mathbf{x}) \chi_r(\mathbf{x};\alpha), \\ \boldsymbol{\rho}^*(\mathbf{x};\alpha) &= \sum_{r=1}^{n} \boldsymbol{\rho}_r^*(\mathbf{x}) \chi_r(\mathbf{x};\alpha), \end{aligned} \tag{72}$$

the ensemble average of the functional (70) is expressed by the following relation:

$$\begin{aligned} &\langle \mathcal{K}[\boldsymbol{\rho}(\mathbf{x}); \boldsymbol{\rho}^*(\mathbf{x})] \rangle \\ &\quad = \sum_{r=1}^{n} c_r \int_{\mathbb{R}^3} \boldsymbol{\rho}_r^*(\mathbf{x}) \left\{ \delta\mathbb{C}_r^{-1} \boldsymbol{\rho}_r(\mathbf{x}) - \mathbf{e}_0(\mathbf{x}) \right\} \mathrm{d}\mathbf{x} \\ &\qquad + \sum_{r,s=1}^{n} \int_{\mathbb{R}^3} \boldsymbol{\rho}_r^*(\mathbf{x}) \\ &\qquad \times \left\{ \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}') \boldsymbol{\rho}_s(\mathbf{x}') P_{rs}(\mathbf{x}-\mathbf{x}') \, \mathrm{d}\mathbf{x}' \right\} \mathrm{d}\mathbf{x}, \end{aligned} \tag{73}$$

which is zero for any arbitrary field $\boldsymbol{\rho}_r^*(\mathbf{x})$ $(r = 1, \dots, n)$ if and only if the following system of integral equations holds:

$$\begin{aligned} c_r \mathbf{e}_0(\mathbf{x}) &= c_r \delta\mathbb{C}_r^{-1} \boldsymbol{\rho}_r(\mathbf{x}) \\ &\quad + \sum_{s=1}^{n} \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}') \boldsymbol{\rho}_s(\mathbf{x}') P_{rs}(\mathbf{x}-\mathbf{x}') \, \mathrm{d}\mathbf{x}'. \end{aligned} \tag{74}$$

[3] The further hypothesis of constant eigenstress $\mathbf{t}_r$ within each phase r is fundamental in solving the integral equation for the polarization stress $\boldsymbol{\rho}$ through Fourier transforming.

Using Eqs. (71) and (72) in (68) and taking the ensemble average of the superimposed deformation field we get:

$$\langle \mathbf{e} \rangle(\mathbf{x}) = \mathbf{e}_0(\mathbf{x}) - \sum_{s=1}^{n} c_s \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}') \boldsymbol{\rho}_s(\mathbf{x}') \, \mathrm{d}\mathbf{x}', \tag{75}$$

which is used together with (74) to get the following system of integral equations for $\boldsymbol{\rho}_r(\mathbf{x})$ $(r = 1, \dots, n)$:

$$\begin{aligned} c_r \langle \mathbf{e} \rangle(\mathbf{x}) &= c_r \delta\mathbb{C}_r^{-1} \boldsymbol{\rho}_r(\mathbf{x}) \\ &\quad + \sum_{s=1}^{n} \int_{\mathbb{R}^3} \boldsymbol{\Gamma}_0(\mathbf{x}-\mathbf{x}') \boldsymbol{\rho}_s(\mathbf{x}') \\ &\quad \times \left[P_{rs}(\mathbf{x}-\mathbf{x}') - c_r c_s \right] \mathrm{d}\mathbf{x}'. \end{aligned} \tag{76}$$

Once the system of integral equations (76) is solved for the unknowns $\boldsymbol{\rho}_r(\mathbf{x})$ $(r = 1, \dots, n)$, the ensemble average

$$\langle \boldsymbol{\rho} \rangle(\mathbf{x}) = \sum_{r=1}^{n} c_r \boldsymbol{\rho}_r(\mathbf{x}) \tag{77}$$

can be obtained and it can ultimately be used to compute the following expression:

$$\langle \mathring{\boldsymbol{\sigma}} \rangle(\mathbf{x}) = \mathbb{L}_0 \langle \mathbf{e} \rangle(\mathbf{x}) + \langle \boldsymbol{\rho} \rangle(\mathbf{x}), \tag{78}$$

namely the average of the incremental constitutive relation (65).

5 Effective non-local constitutive equations

From the previous sections we may notice that the different choices of polarizations $\boldsymbol{\tau}(\mathbf{x})$, $\mathbf{p}(\mathbf{x})$, $\mathbf{q}(\mathbf{x})$ and $\boldsymbol{\rho}(\mathbf{x})$ are related to one another. This is summarized in the following list of equations:

$$\begin{aligned} \mathbf{p}(\mathbf{x}) &= \mathbf{q}(\mathbf{x}) + \boldsymbol{\Sigma}_0(\mathbf{x}) = \boldsymbol{\tau}(\mathbf{x}) + \boldsymbol{\Sigma}(\mathbf{x}) \\ &= \boldsymbol{\rho}(\mathbf{x}) + \mathbf{t}(\mathbf{x}) + \mathbf{w}(\mathbf{x})\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x})\mathbf{w}(\mathbf{x}). \end{aligned} \tag{79}$$

In the sequel, attention will be focused on two-phase composites in order to evaluate their effective response according to the three total formulations ($\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$) and the incremental approach ($\mathcal{B}$) previously introduced.

5.1 Two-phase composites

Restricting now, for simplicity, attention to the case of two-phase composites, we have that

$$P_{rs}\big(\mathbf{x}-\mathbf{x}'\big)-c_rc_s=c_r(\delta_{rs}-c_s)h\big(\mathbf{x}-\mathbf{x}'\big)\quad\text{no sum,}\tag{80}$$

where $h(\mathbf{x}-\mathbf{x}')$ is the two-point correlation function [28], the integral equations systems (36), (49), (62), (76) can be rewritten respectively as follows ($r=1,2$):

$$\begin{aligned}
&(\mathcal{A}_1)\\
&c_r\langle\mathbf{e}\rangle(\mathbf{x})=c_r\delta\mathbb{L}_r^{-1}\boldsymbol{\tau}_r(\mathbf{x})+c_r\sum_{s=1}^{2}(\delta_{rs}-c_s)\\
&\qquad\times\int_{\mathbb{R}^3}\boldsymbol{\Upsilon}_0\big(\mathbf{x}-\mathbf{x}'\big)\big[\boldsymbol{\tau}_s\big(\mathbf{x}'\big)+\boldsymbol{\Sigma}_s\big(\mathbf{x}'\big)\big]\,d\mathbf{x}',\\
&(\mathcal{A}_2)\\
&c_r\langle\mathbf{e}\rangle(\mathbf{x})=c_r\delta\mathbb{L}_r^{-1}\big[\mathbf{p}_r(\mathbf{x})-\boldsymbol{\Sigma}_r(\mathbf{x})\big]\\
&\qquad+c_r\sum_{s=1}^{2}(\delta_{rs}-c_s)\\
&\qquad\times\int_{\mathbb{R}^3}\boldsymbol{\Upsilon}_0\big(\mathbf{x}-\mathbf{x}'\big)\mathbf{p}_s\big(\mathbf{x}'\big)\,d\mathbf{x}',\\
&(\mathcal{A}_3)\\
&c_r\langle\mathbf{e}\rangle(\mathbf{x})=c_r\delta\mathbb{L}_r^{-1}\big[\mathbf{q}_r(\mathbf{x})+\boldsymbol{\Sigma}_0(\mathbf{x})-\boldsymbol{\Sigma}_r(\mathbf{x})\big]\\
&\qquad+c_r\sum_{s=1}^{2}(\delta_{rs}-c_s)\\
&\qquad\times\int_{\mathbb{R}^3}\boldsymbol{\Upsilon}_0\big(\mathbf{x}-\mathbf{x}'\big)\mathbf{q}_s\big(\mathbf{x}'\big)\,d\mathbf{x}',\\
&(\mathcal{B})\\
&c_r\langle\mathbf{e}\rangle(\mathbf{x})=c_r\delta\mathbb{C}_r^{-1}\boldsymbol{\rho}_r(\mathbf{x})+c_r\sum_{s=1}^{2}(\delta_{rs}-c_s)\\
&\qquad\times\int_{\mathbb{R}^3}\boldsymbol{\Upsilon}_0\big(\mathbf{x}-\mathbf{x}'\big)\boldsymbol{\rho}_s\big(\mathbf{x}'\big)\,d\mathbf{x}',
\end{aligned}\tag{81}$$

after setting

$$\boldsymbol{\Upsilon}_0(\mathbf{x})=\boldsymbol{\Gamma}_0(\mathbf{x})h(\mathbf{x}).\tag{82}$$

Fourier transforms of the previous equations are considered in the sequel. To this end, the following form of the three-dimensional Fourier transform $\widetilde{f}(\boldsymbol{\xi})$ of any function $f(\mathbf{x})$ and its inverse are defined as follows:

$$\begin{aligned}
&\widetilde{f}(\boldsymbol{\xi})=\int_{\mathbb{R}^3}f(\mathbf{x})e^{\imath\boldsymbol{\xi}\cdot\mathbf{x}}\,d\mathbf{x},\\
&f(\mathbf{x})=\frac{1}{8\pi^3}\int_{\mathbb{R}^3}\widetilde{f}(\boldsymbol{\xi})e^{-\imath\boldsymbol{\xi}\cdot\mathbf{x}}\,d\boldsymbol{\xi},
\end{aligned}\tag{83}$$

where $\imath$ is the imaginary unit, $\boldsymbol{\xi}$ is the position in the transformed space and $\cdot$ represents the scalar product between vectors.

Computing such transform for (81) leads to the following expressions ($r=1,2$):

$$\begin{aligned}
&(\mathcal{A}_1)\\
&c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi})=c_r\delta\mathbb{L}_r^{-1}\widetilde{\boldsymbol{\tau}}_r(\boldsymbol{\xi})\\
&\qquad+c_r\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})\sum_{s=1}^{2}(\delta_{rs}-c_s)\big[\widetilde{\boldsymbol{\tau}}_s(\boldsymbol{\xi})+\widetilde{\boldsymbol{\Sigma}}_s(\boldsymbol{\xi})\big],\\
&(\mathcal{A}_2)\\
&c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi})=c_r\delta\mathbb{L}_r^{-1}\big[\widetilde{\mathbf{p}}_r(\boldsymbol{\xi})-\widetilde{\boldsymbol{\Sigma}}_r(\boldsymbol{\xi})\big]\\
&\qquad+c_r\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})\sum_{s=1}^{2}(\delta_{rs}-c_s)\widetilde{\mathbf{p}}_s(\boldsymbol{\xi}),\\
&(\mathcal{A}_3)\\
&c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi})=c_r\delta\mathbb{L}_r^{-1}\big[\widetilde{\mathbf{q}}_r(\boldsymbol{\xi})+\widetilde{\boldsymbol{\Sigma}}_0(\boldsymbol{\xi})-\widetilde{\boldsymbol{\Sigma}}_r(\boldsymbol{\xi})\big]\\
&\qquad+c_r\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})\sum_{s=1}^{2}(\delta_{rs}-c_s)\widetilde{\mathbf{q}}_s(\boldsymbol{\xi}),\\
&(\mathcal{B})\\
&c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi})=c_r\delta\mathbb{C}_r^{-1}\widetilde{\boldsymbol{\rho}}_r(\boldsymbol{\xi})\\
&\qquad+c_r\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})\sum_{s=1}^{2}(\delta_{rs}-c_s)\widetilde{\boldsymbol{\rho}}_s(\boldsymbol{\xi}),
\end{aligned}\tag{84}$$

where

$$\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})=(\widetilde{\boldsymbol{\Gamma}}_0*\widetilde{h})(\boldsymbol{\xi})=\frac{1}{8\pi^3}\int_{\mathbb{R}^3}\widetilde{\boldsymbol{\Gamma}}_0\big(\boldsymbol{\xi}-\boldsymbol{\xi}'\big)\widetilde{h}\big(\boldsymbol{\xi}'\big)\,d\boldsymbol{\xi}'.\tag{85}$$

Upon introducing the ‘stiffness-like’ operators

$$\begin{aligned}
&\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})=\delta\mathbb{L}_r\widetilde{\mathbb{K}}(\boldsymbol{\xi})\widetilde{\mathbb{H}}_{rs}(\boldsymbol{\xi}),\\
&\widetilde{\mathbb{T}}^*_{rs}(\boldsymbol{\xi})=\delta\mathbb{C}_r\widetilde{\mathbb{K}}^*(\boldsymbol{\xi})\widetilde{\mathbb{H}}^*_{rs}(\boldsymbol{\xi}),
\end{aligned}\tag{86}$$

where

$$\begin{aligned}
&\widetilde{\mathbb{K}}(\boldsymbol{\xi})=\big[\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\boldsymbol{\xi})+c_1\delta\mathbb{L}_2+(1-c_1)\delta\mathbb{L}_1\big]^{-1},\\
&\widetilde{\mathbb{H}}_{rs}(\boldsymbol{\xi})=\frac{\delta_{rs}}{c_s}\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\boldsymbol{\xi})+\delta\mathbb{L}_1+\delta\mathbb{L}_2-\delta\mathbb{L}_r,\\
&\widetilde{\mathbb{K}}^*(\boldsymbol{\xi})=\big[\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\boldsymbol{\xi})+c_1\delta\mathbb{C}_2+(1-c_1)\delta\mathbb{C}_1\big]^{-1},\\
&\widetilde{\mathbb{H}}^*_{rs}(\boldsymbol{\xi})=\frac{\delta_{rs}}{c_s}\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\boldsymbol{\xi})+\delta\mathbb{C}_1+\delta\mathbb{C}_2-\delta\mathbb{C}_r,
\end{aligned}\tag{87}$$

and their inverse

$$\widetilde{\mathbb{T}}_{rs}^{-1}(\boldsymbol{\xi}) = c_r \delta\mathbb{L}_r^{-1}\delta_{rs} + c_r(\delta_{rs} - c_s)\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi}),$$
$$\left(\widetilde{\mathbb{T}}_{rs}^{*}\right)^{-1}(\boldsymbol{\xi}) = c_r \delta\mathbb{C}_r^{-1}\delta_{rs} + c_r(\delta_{rs} - c_s)\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi}), \tag{88}$$

the systems (81) take the form ($r = 1, 2$)

$$(\mathcal{A}_1)\quad \sum_{s=1}^{2} \widetilde{\mathbb{T}}_{rs}^{-1}(\boldsymbol{\xi})\widetilde{\boldsymbol{\tau}}_s(\boldsymbol{\xi}) = c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) - c_r\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})\sum_{z=1}^{2}(\delta_{rz} - c_z)\widetilde{\boldsymbol{\Sigma}}_z(\boldsymbol{\xi}),$$
$$(\mathcal{A}_2)\quad \sum_{s=1}^{2} \widetilde{\mathbb{T}}_{rs}^{-1}(\boldsymbol{\xi})\widetilde{\mathbf{p}}_s(\boldsymbol{\xi}) = c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + c_r\delta\mathbb{L}_r^{-1}\widetilde{\boldsymbol{\Sigma}}_r(\boldsymbol{\xi}), \tag{89}$$
$$(\mathcal{A}_3)\quad \sum_{s=1}^{2} \widetilde{\mathbb{T}}_{rs}^{-1}(\boldsymbol{\xi})\widetilde{\mathbf{q}}_s(\boldsymbol{\xi}) = c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + c_r\delta\mathbb{L}_r^{-1}\left[\widetilde{\boldsymbol{\Sigma}}_r(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_0(\boldsymbol{\xi})\right],$$
$$(\mathcal{B})\quad \sum_{s=1}^{2}\left(\widetilde{\mathbb{T}}_{rs}^{*}\right)^{-1}(\boldsymbol{\xi})\widetilde{\boldsymbol{\rho}}_s(\boldsymbol{\xi}) = c_r\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}).$$

Solutions for such systems of equations may be easily provided as follows ($r = 1, 2$):

$$(\mathcal{A}_1)\quad \widetilde{\boldsymbol{\tau}}_r(\boldsymbol{\xi}) = \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) - \sum_{s,z=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})(\delta_{sz} - c_z)\widetilde{\boldsymbol{\Sigma}}_z(\boldsymbol{\xi}),$$
$$(\mathcal{A}_2)\quad \widetilde{\mathbf{p}}_r(\boldsymbol{\xi}) = \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\delta\mathbb{L}_s^{-1}\widetilde{\boldsymbol{\Sigma}}_s(\boldsymbol{\xi}), \tag{90}$$
$$(\mathcal{A}_3)\quad \widetilde{\mathbf{q}}_r(\boldsymbol{\xi}) = \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\delta\mathbb{L}_s^{-1}\left[\widetilde{\boldsymbol{\Sigma}}_s(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_0(\boldsymbol{\xi})\right],$$
$$(\mathcal{B})\quad \widetilde{\boldsymbol{\rho}}_r(\boldsymbol{\xi}) = \sum_{s=1}^{2} c_s\widetilde{\mathbb{T}}_{rs}^{*}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}).$$

Finally, from the Fourier transform of the stress polarization fields (90) we are able to obtain their ensemble averages through the Fourier Transform of Eqs. (37), (50), (63), (77), i.e.

$$(\mathcal{A}_1)\quad \langle\widetilde{\boldsymbol{\tau}}\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) - \sum_{r,s,z=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\widetilde{\boldsymbol{\Upsilon}}_0(\boldsymbol{\xi})(\delta_{sz} - c_z)\widetilde{\boldsymbol{\Sigma}}_z(\boldsymbol{\xi}),$$
$$(\mathcal{A}_2)\quad \langle\widetilde{\mathbf{p}}\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\delta\mathbb{L}_s^{-1}\widetilde{\boldsymbol{\Sigma}}_s(\boldsymbol{\xi}), \tag{91}$$
$$(\mathcal{A}_3)\quad \langle\widetilde{\mathbf{q}}\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi})\delta\mathbb{L}_s^{-1}\left[\widetilde{\boldsymbol{\Sigma}}_s(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_0(\boldsymbol{\xi})\right],$$
$$(\mathcal{B})\quad \langle\widetilde{\boldsymbol{\rho}}\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}^{*}(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}).$$

Upon evaluating the ensemble averages of the ‘stiffness-like’ operators

$$\langle\widetilde{\mathbb{T}}\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}(\boldsymbol{\xi}),$$
$$\left\langle\widetilde{\mathbb{T}}^{*}\right\rangle(\boldsymbol{\xi}) = \sum_{r,s=1}^{2} c_r c_s\widetilde{\mathbb{T}}_{rs}^{*}(\boldsymbol{\xi}), \tag{92}$$

evaluating the dimensionless tensor:

$$\widetilde{\mathbb{S}}(\boldsymbol{\xi}) = c_1(1 - c_1)(\mathbb{L}_1 - \mathbb{L}_2)\widetilde{\mathbb{K}}(\boldsymbol{\xi}), \tag{93}$$

and the stress-like variables

$$\widetilde{\widetilde{\boldsymbol{\Sigma}}}_1(\boldsymbol{\xi}) = \mathbb{L}_1\mathbb{L}_0^{-1}\boldsymbol{\Sigma}_1(\boldsymbol{\xi}),$$
$$\widetilde{\widetilde{\boldsymbol{\Sigma}}}_2(\boldsymbol{\xi}) = \mathbb{L}_2\mathbb{L}_0^{-1}\boldsymbol{\Sigma}_2(\boldsymbol{\xi}), \tag{94}$$

Eqs. (91) can be rewritten as follows:

$$(\mathcal{A}_1)\quad \langle\widetilde{\boldsymbol{\tau}}\rangle(\boldsymbol{\xi}) = \langle\widetilde{\mathbb{T}}\rangle(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) - \widetilde{\mathbb{S}}(\boldsymbol{\xi})\left[\widetilde{\widetilde{\boldsymbol{\Sigma}}}_1(\boldsymbol{\xi}) - \widetilde{\widetilde{\boldsymbol{\Sigma}}}_2(\boldsymbol{\xi})\right],$$

$$
\begin{aligned}
&(\mathcal{A}_2)\\
&\langle\widetilde{\mathbf{p}}\rangle(\boldsymbol{\xi}) = \langle\widetilde{\mathbb{T}}\rangle(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \langle\widetilde{\boldsymbol{\Sigma}}\rangle(\boldsymbol{\xi})\\
&\qquad - \widetilde{\mathbb{S}}(\boldsymbol{\xi})\big[\widetilde{\boldsymbol{\Sigma}}_1(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_2(\boldsymbol{\xi})\big],\\
&(\mathcal{A}_3)\\
&\langle\widetilde{\mathbf{q}}\rangle(\boldsymbol{\xi}) = \langle\widetilde{\mathbb{T}}\rangle(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}) + \langle\widetilde{\boldsymbol{\Sigma}}\rangle(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_0(\boldsymbol{\xi})\\
&\qquad - \widetilde{\mathbb{S}}(\boldsymbol{\xi})\big[\widetilde{\boldsymbol{\Sigma}}_1(\boldsymbol{\xi}) - \widetilde{\boldsymbol{\Sigma}}_2(\boldsymbol{\xi})\big],\\
&(\mathcal{B})\\
&\langle\widetilde{\boldsymbol{\rho}}\rangle(\boldsymbol{\xi}) = \big\langle\widetilde{\mathbb{T}}^*\big\rangle(\boldsymbol{\xi})\langle\widetilde{\mathbf{e}}\rangle(\boldsymbol{\xi}).
\end{aligned}
\tag{95}
$$

After simple manipulations, we conclude that the effective constitutive models arising from the three total formulations, Eqs. (38), (51), (64), are actually equivalent, i.e.

$$(\mathcal{A}_1) \equiv (\mathcal{A}_2) \equiv (\mathcal{A}_3).$$

This important point means that the result is not affected by the choices of comparison solids considered before, in other words by letting:

- the residual stress in the real material to prestress the comparison solid, so the polarization does not contain any residual stress field;
- the comparison medium to be completely un-prestressed, so that the corresponding polarization contains the whole residual stress;
- the comparison solid to undergo an arbitrary prestress, which may not necessarily coincide with the residual stress existing in the heterogeneous medium, so that the corresponding polarization contains the difference between the real and the arbitrary prestress;

lead to the same result.

In the sequel we then shall refer to these three formulations with the label $\mathcal{A}$.

Inverse Fourier transforms of (95), and the convolution theorem, may be considered in order to evaluate the spatial distributions of the stress polarizations associated with the remaining two formulations $\mathcal{A}$ and $\mathcal{B}$, i.e.:

$$
\begin{aligned}
&(\mathcal{A})\\
&\langle\boldsymbol{\tau}\rangle(\mathbf{x}) = \int_{\mathbb{R}^3}\big\{\langle\mathbb{T}\rangle\big(\mathbf{x}-\mathbf{x}'\big)\langle\mathbf{e}\rangle\big(\mathbf{x}'\big)\\
&\qquad - \mathbb{S}\big(\mathbf{x}-\mathbf{x}'\big)\big[\boldsymbol{\Sigma}_1\big(\mathbf{x}'\big) - \boldsymbol{\Sigma}_2\big(\mathbf{x}'\big)\big]\big\}\,\mathrm{d}\mathbf{x}',\\
&(\mathcal{B})\\
&\langle\boldsymbol{\rho}\rangle(\mathbf{x}) = \int_{\mathbb{R}^3}\big\langle\mathbb{T}^*\big\rangle\big(\mathbf{x}-\mathbf{x}'\big)\langle\mathbf{e}\rangle\big(\mathbf{x}'\big)\,\mathrm{d}\mathbf{x}'.
\end{aligned}
\tag{96}
$$

Approximating $\langle\mathbf{e}\rangle(\mathbf{x}')$ and $\boldsymbol{\Sigma}_k(\mathbf{x}')$ $(k=1,2)$ by the first three-terms of their Taylor expansions in the neighborhood of the position $\mathbf{x}$ may be considered as follows:

$$
\begin{aligned}
\langle\mathbf{e}\rangle\big(\mathbf{x}'\big) &\simeq \langle\mathbf{e}\rangle(\mathbf{x}) + \big(\mathbf{x}'-\mathbf{x}\big)\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\
&\quad + \frac{1}{2}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x}),\\
\boldsymbol{\Sigma}_k\big(\mathbf{x}'\big) &\simeq \boldsymbol{\Sigma}_k(\mathbf{x}) + \big(\mathbf{x}'-\mathbf{x}\big)\nabla\boldsymbol{\Sigma}_k(\mathbf{x})\\
&\quad + \frac{1}{2}\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\nabla\nabla\boldsymbol{\Sigma}_k(\mathbf{x}),
\end{aligned}
\tag{97}
$$

so that polarization fields (96) admit the following forms:

$$
\begin{aligned}
&(\mathcal{A})\\
&\langle\boldsymbol{\tau}\rangle(\mathbf{x}) \simeq \bigg[\int_{\mathbb{R}^3}\langle\mathbb{T}\rangle\big(\mathbf{x}-\mathbf{x}'\big)\,\mathrm{d}\mathbf{x}'\bigg]\langle\mathbf{e}\rangle(\mathbf{x})\\
&\qquad + \bigg[\int_{\mathbb{R}^3}\langle\mathbb{T}\rangle\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\
&\qquad + \frac{1}{2}\bigg[\int_{\mathbb{R}^3}\langle\mathbb{T}\rangle\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\\
&\qquad \times \nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\
&\qquad - \bigg[\int_{\mathbb{R}^3}\mathbb{S}\big(\mathbf{x}-\mathbf{x}'\big)\,\mathrm{d}\mathbf{x}'\bigg]\big[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\big]\\
&\qquad - \bigg[\int_{\mathbb{R}^3}\mathbb{S}\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\nabla\\
&\qquad \times \big[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\big]\\
&\qquad - \frac{1}{2}\bigg[\int_{\mathbb{R}^3}\mathbb{S}\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\\
&\qquad \times \nabla\nabla\big[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\big],\\
&(\mathcal{B})\\
&\langle\boldsymbol{\rho}\rangle(\mathbf{x}) \simeq \bigg[\int_{\mathbb{R}^3}\big\langle\mathbb{T}^*\big\rangle\big(\mathbf{x}-\mathbf{x}'\big)\mathrm{d}\mathbf{x}'\bigg]\langle\mathbf{e}\rangle(\mathbf{x})\\
&\qquad + \bigg[\int_{\mathbb{R}^3}\big\langle\mathbb{T}^*\big\rangle\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\
&\qquad + \frac{1}{2}\bigg[\int_{\mathbb{R}^3}\big\langle\mathbb{T}^*\big\rangle\big(\mathbf{x}-\mathbf{x}'\big)\big(\mathbf{x}'-\mathbf{x}\big)\big(\mathbf{x}'-\mathbf{x}\big)\,\mathrm{d}\mathbf{x}'\bigg]\\
&\qquad \times \nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x}).
\end{aligned}
\tag{98}
$$

As observed in [15], convolution integrals in the whole space and their first moments can be simplified by making use of the following identities:

$$\int_{\mathbb{R}^3} f\left(\mathbf{x}-\mathbf{x}'\right)\mathrm{d}\mathbf{x}' = \widetilde{f}(\boldsymbol{\xi}=\mathbf{0}),$$
$$\int_{\mathbb{R}^3} f\left(\mathbf{x}-\mathbf{x}'\right)\left(\mathbf{x}'-\mathbf{x}\right)\mathrm{d}\mathbf{x}' = \imath\nabla_\xi \widetilde{f}(\boldsymbol{\xi}=\mathbf{0}), \tag{99}$$
$$\int_{\mathbb{R}^3} f\left(\mathbf{x}-\mathbf{x}'\right)\left(\mathbf{x}'-\mathbf{x}\right)\left(\mathbf{x}'-\mathbf{x}\right)\mathrm{d}\mathbf{x}' = -\nabla_\xi\nabla_\xi \widetilde{f}(\boldsymbol{\xi}=\mathbf{0}).$$

5.2 Local response to homogeneous prestress and superimposed strain fields

If the prestress and superimposed strain are constant fields, the composite obviously responds with local terms only, namely:

$$\begin{aligned}(\mathcal{A})\quad &\langle\boldsymbol{\sigma}\rangle = \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle\\ &\qquad + \langle\boldsymbol{\Sigma}\rangle - \widetilde{\mathbb{S}}(\mathbf{0})(\boldsymbol{\Sigma}_1-\boldsymbol{\Sigma}_2),\\ (\mathcal{B})\quad &\langle\overset{\circ}{\boldsymbol{\sigma}}\rangle = \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle.\end{aligned} \tag{100}$$

This situation is of great interest since it allows for highlighting the local part of the overall response of the composite and, as in [15], may be utilized as a benchmark against approximate expressions entailing second gradient terms for the effective constitutive equation of the composite when spatially nonconstant fields are considered.

5.3 Isotropic phase distribution

Isotropic distributions of the phases, possible anisotropic in terms of their constitutive behavior, are considered in the sequel. Under this restriction, it is known from [15] that the two-point correlation function $h(\mathbf{x})$ satisfies

$$h(\mathbf{x}) = h\left(|\mathbf{x}|\right) \quad \Rightarrow \quad \widetilde{h}(\boldsymbol{\xi}) = \widetilde{h}\left(|\boldsymbol{\xi}|\right). \tag{101}$$

Through this property, Drugan and Willis in [15] have shown that

$$\begin{aligned}&\widetilde{\boldsymbol{\Upsilon}}_0(\mathbf{0}) = \frac{1}{4\pi}\int_{|\boldsymbol{\xi}|=1}\widetilde{\boldsymbol{\Gamma}}_0(\boldsymbol{\xi})\,\mathrm{d}S,\\ &\frac{\partial\widetilde{\boldsymbol{\Upsilon}}_0}{\partial\xi_m}(\mathbf{0}) = \mathbf{0},\\ &\frac{\partial^2\widetilde{\boldsymbol{\Upsilon}}_0}{\partial\xi_m\partial\xi_n}(\mathbf{0}) = \frac{1}{4\pi}\int_{|\boldsymbol{\xi}|=1}(3\xi_m\xi_n-\delta_{mn})\widetilde{\boldsymbol{\Gamma}}_0(\boldsymbol{\xi})\,\mathrm{d}S\\ &\qquad\times\int_0^\infty h(r)r\,\mathrm{d}r,\end{aligned} \tag{102}$$

and therefore,

$$\begin{aligned}&\langle\widetilde{\mathbb{T}}\rangle_{,m}(\mathbf{0}) = \langle\widetilde{\mathbb{T}}^*\rangle_{,m}(\mathbf{0}) = \widetilde{\mathbb{S}}_{,m}(\mathbf{0}) = \mathbf{0},\\ &\langle\widetilde{\mathbb{T}}\rangle_{,mn}(\mathbf{0}) = -c_1(1-c_1)(\delta\mathbb{L}_1-\delta\mathbb{L}_2),\\ &\widetilde{\mathbb{K}}(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_{0,mn}(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\mathbf{0})\widetilde{\mathbb{K}}(\mathbf{0})(\delta\mathbb{L}_1-\delta\mathbb{L}_2),\\ &\langle\widetilde{\mathbb{T}}^*\rangle_{,mn}(\mathbf{0}) = -c_1(1-c_1)(\delta\mathbb{C}_1-\delta\mathbb{C}_2),\\ &\widetilde{\mathbb{K}}^*(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_{0,mn}(\mathbf{0})\widetilde{\boldsymbol{\Upsilon}}_0^{-1}(\mathbf{0})\widetilde{\mathbb{K}}^*(\mathbf{0})(\delta\mathbb{C}_1-\delta\mathbb{C}_2),\\ &\widetilde{\mathbb{S}}_{,mn}(\mathbf{0}) = -\langle\widetilde{\mathbb{T}}\rangle_{,mn}(\mathbf{0})(\delta\mathbb{L}_1-\delta\mathbb{L}_2)^{-1}.\end{aligned} \tag{103}$$

Because the quantities $\langle\widetilde{\mathbb{T}}\rangle_{,m}(\mathbf{0})$, $\langle\widetilde{\mathbb{T}}^*\rangle_{,m}(\mathbf{0})$ and $\widetilde{\mathbb{S}}_{,m}(\mathbf{0})$ vanish, the approximation for the polarization stress (98) simplifies as

$$\begin{aligned}&(\mathcal{A})\\ &\langle\boldsymbol{\tau}\rangle(\mathbf{x}) \simeq \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\langle\mathbf{e}\rangle(\mathbf{x}) - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\ &\qquad - \widetilde{\mathbb{S}}(\mathbf{0})\left[\boldsymbol{\Sigma}_1(\mathbf{x})-\boldsymbol{\Sigma}_2(\mathbf{x})\right]\\ &\qquad + \frac{1}{2}\nabla_\xi\nabla_\xi\widetilde{\mathbb{S}}(\mathbf{0})\nabla\nabla\left[\boldsymbol{\Sigma}_1(\mathbf{x})-\boldsymbol{\Sigma}_2(\mathbf{x})\right],\\ &(\mathcal{B})\\ &\langle\boldsymbol{\rho}\rangle(\mathbf{x}) \simeq \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\langle\mathbf{e}\rangle(\mathbf{x}) - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x}),\end{aligned} \tag{104}$$

so that the effective responses (38) and (78) turn out to be approximated by the following expressions:

$$\begin{aligned}&(\mathcal{A})\\ &\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) \simeq \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle(\mathbf{x})\\ &\qquad - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x})\\ &\qquad + \langle\boldsymbol{\Sigma}\rangle(\mathbf{x}) - \widetilde{\mathbb{S}}(\mathbf{0})\left[\boldsymbol{\Sigma}_1(\mathbf{x})-\boldsymbol{\Sigma}_2(\mathbf{x})\right]\\ &\qquad + \frac{1}{2}\nabla_\xi\nabla_\xi\widetilde{\mathbb{S}}(\mathbf{0})\nabla\nabla\left[\boldsymbol{\Sigma}_1(\mathbf{x})-\boldsymbol{\Sigma}_2(\mathbf{x})\right],\\ &(\mathcal{B})\\ &\langle\overset{\circ}{\boldsymbol{\sigma}}\rangle(\mathbf{x}) \simeq \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle(\mathbf{x})\\ &\qquad - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x}).\end{aligned} \tag{105}$$

The comparison medium is chosen as usual to be coincident with the matrix in which the random inclusions are contained, namely

$$\mathbb{L}_0 = \mathbb{L}_2. \tag{106}$$

It follows that the dimensionless tensor $\widetilde{\mathbb{S}}(\mathbf{0})$ appearing in Eq. (93) can be related to $\langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})$ in a simple way

$$\widetilde{\mathbb{S}}(\mathbf{0}) = c_1\widetilde{\mathbb{I}} - \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})(\mathbb{L}_1-\mathbb{L}_2)^{-1}, \tag{107}$$

so that its second derivative simplifies as follows

$$\widetilde{\mathbb{S}}_{,mn}(\mathbf{0}) = -\langle\widetilde{\mathbb{T}}\rangle_{,mn}(\mathbf{0})(\mathbb{L}_1 - \mathbb{L}_2)^{-1}. \tag{108}$$

Taking into account all the simplified expressions above, the non-local constitutive equations (105) may be rewritten in the forms:

$$\begin{aligned}
&(\mathcal{A})\\
&\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) \simeq \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle(\mathbf{x}) + \boldsymbol{\Sigma}_2(\mathbf{x})\\
&\quad + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})(\mathbb{L}_1 - \mathbb{L}_2)^{-1}\left[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\right]\\
&\quad - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\left\{\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x})\right.\\
&\quad \left. + (\mathbb{L}_1 - \mathbb{L}_2)^{-1}\nabla\nabla\left[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\right]\right\},\\
&(\mathcal{B})\\
&\langle\overset{\circ}{\boldsymbol{\sigma}}\rangle(\mathbf{x}) \simeq \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\right]\langle\mathbf{e}\rangle(\mathbf{x})\\
&\quad - \frac{1}{2}\nabla_\xi\nabla_\xi\langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0})\nabla\nabla\langle\mathbf{e}\rangle(\mathbf{x}).
\end{aligned} \tag{109}$$

Equations (109) yield both the total and the incremental forms of the non-local constitutive relations for a two-phase composite with isotropic phase distributions in the presence of a pre-existing stress state.

In analogy with [15], from such relations we note that while there is the dependence on the average of the superimposed strain $\langle\mathbf{e}\rangle(\mathbf{x})$ and its second derivative for both the total and the incremental approach, the dependence on the prestress in the total formulation is given in terms of the difference $\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})$ and its second derivative.

5.4 Isotropic phases

In the sequel we particularize the obtained results to composites containing isotropic phases, i.e.

$$\begin{aligned}
\mathbb{L}_{1\,ijkl} &= \left(\kappa_1 - \frac{2}{3}\mu_1\right)\delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),\\
\mathbb{L}_{2\,ijkl} &= \left(\kappa_2 - \frac{2}{3}\mu_2\right)\delta_{ij}\delta_{kl} + \mu_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).
\end{aligned} \tag{110}$$

In this case, the components of $\langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})$ and its second gradient are the following (see [15]):

$$\begin{aligned}
&\langle\widetilde{\mathbb{T}}\rangle_{ijkl}(\mathbf{0})\\
&\quad = \frac{c_1(\kappa_1 - \kappa_2)(3\kappa_2 + 4\mu_2)}{3\kappa_1 + 4\mu_2 - 3c_1(\kappa_1 - \kappa_2)}\delta_{ij}\delta_{kl}\\
&\qquad + \frac{5c_1\mu_2(\mu_1 - \mu_2)(3\kappa_2 + 4\mu_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{5\mu_2(3\kappa_2 + 4\mu_2) + 6(1 - c)(\mu_1 - \mu_2)(\kappa_2 + 2\mu_2)},
\end{aligned}$$

$$\begin{aligned}
&\langle\widetilde{\mathbb{T}}\rangle_{ijkl,mn}(\mathbf{0})\\
&\quad = -c_1(1 - c_1)H\\
&\qquad \times \left\{B_1\delta_{ij}\delta_{kl}\delta_{mn} + B_2(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn})\right.\\
&\qquad + B_3(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{im}\delta_{jn}\delta_{kl}\\
&\qquad + \delta_{in}\delta_{jm}\delta_{kl}) + B_4\big(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}\\
&\qquad + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jk}\delta_{ln}\\
&\qquad \left. + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{in}\delta_{jl}\delta_{km}\big)\right\},
\end{aligned} \tag{111}$$

where

$$\begin{aligned}
B_1 &= \frac{4}{3}(3A_1 + 2A_2 + 2A_3)(3\kappa_B - 2\mu_B)\mu_B\\
&\quad + 4A_1\mu_B^2,\\
B_2 &= 4A_2\mu_B^2, \qquad B_3 = -\frac{3}{4}B_1, \qquad B_4 = -\frac{3}{4}B_2,\\
A_1 &= \frac{4}{105}\frac{3\kappa_0 + \mu_0}{\mu_0(3\kappa_0 + 4\mu_0)},\\
A_2 &= -\frac{1}{35}\frac{3\kappa_0 + 8\mu_0}{\mu_0(3\kappa_0 + 4\mu_0)},\\
A_3 &= -\frac{3}{4}A_1, \qquad A_4 = -\frac{3}{4}A_2,\\
\kappa_B &= \frac{(\kappa_1 - \kappa_2)(3\kappa_2 + 4\mu_2)}{3\kappa_2 + 4\mu_2 + 3(1 - c_1)(\kappa_1 - \kappa_2)}\\
\mu_B &= \frac{5\mu_2(\mu_1 - \mu_2)(3\kappa_2 + 4\mu_2)}{5\mu_2(3\kappa_2 + 4\mu_2) + 6(1 - c_1)(\mu_1 - \mu_2)(\kappa_2 + 2\mu_2)},
\end{aligned} \tag{112}$$

and

$$H = \int_0^\infty h(r)\,\mathrm{d}r, \tag{113}$$

so that in the case of nonoverlapping identical spherical inclusions the latter reads as follows:

$$H = a^2\frac{(2 - c_1)(1 - c_1)}{5(1 + 2c_1)}. \tag{114}$$

Finally, in the case of isotropic material, the sixth-order tensor $\mathcal{D}$ can be expressed as follows (see [16]):

$$\begin{aligned}
\mathcal{D}[\mathbf{t}, \mathbf{e}] &= \beta^{(1)}(\mathrm{tr}\,\mathbf{e})(\mathrm{tr}\,\mathbf{t})\mathbf{I} + \beta^{(2)}(\mathrm{tr}\,\mathbf{t})\mathbf{e}\\
&\quad + \beta^{(3)}\left\{(\mathrm{tr}\,\mathbf{e})\mathbf{t} + \left[\mathrm{tr}(\mathbf{et})\right]\mathbf{I}\right\} + \beta^{(4)}[\mathbf{et} + \mathbf{te}],
\end{aligned} \tag{115}$$

where $\beta^{(j)}$ $(j = 1, \dots, 4)$ represent material dimensionless constants.

6 Quantitative estimates of minimum RVE size

In analogy with [15], this section is devoted to obtain quantitative estimates of the minimum size ℓ of the RVE (Representative Volume Element) required to approximate the second order nonlocal response of the prestressed solid with its local part (100) within a maximum fixed small discrepancy. In other words, a maximum fixed error must not be exceeded whenever the local effective response is considered to approximate the overall constitutive behavior of the random composite in comparison to the full second gradient nonlocal description (109).

In order to achieve such minimum size we compare the non-local response originated by superimposition of a sinusoidal strain field upon a prestress field to the local part of the overall constitutive equation for the random composite, namely:

$$\begin{aligned}
&(\mathcal{A})\\
&\langle \boldsymbol{\sigma} \rangle(\mathbf{x}) = \big[\mathbb{L}_0 + \langle \widetilde{\mathbb{T}} \rangle(\mathbf{0})\big] \langle \mathbf{e} \rangle(\mathbf{x}) + \boldsymbol{\Sigma}_2(\mathbf{x})\\
&\qquad + \langle \widetilde{\mathbb{T}} \rangle(\mathbf{0})(\mathbb{L}_1 - \mathbb{L}_2)^{-1}\big[\boldsymbol{\Sigma}_1(\mathbf{x}) - \boldsymbol{\Sigma}_2(\mathbf{x})\big],\\
&(\mathcal{B})\\
&\langle \mathring{\boldsymbol{\sigma}} \rangle(\mathbf{x}) = \big[\mathbb{L}_0 + \langle \widetilde{\mathbb{T}}^* \rangle(\mathbf{0})\big] \langle \mathbf{e} \rangle(\mathbf{x}),
\end{aligned} \tag{116}$$

where, for the total approach $(\mathcal{A})$ the local part of relation (109) has been recast in a slightly more revealing way.

In particular, we examine the following cases:

- deformable (and void) inclusions

$$\mathbf{e}_1(\mathbf{x}) = \mathbf{e}_2(\mathbf{x}) = \langle \mathbf{e} \rangle(\mathbf{x}) = \bar{\mathbf{e}} \sin\left(\frac{2\pi x_1}{\ell}\right), \tag{117}$$

- rigid inclusions

$$\mathbf{e}_1(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{e}_2(\mathbf{x}) = \frac{\langle \mathbf{e} \rangle(\mathbf{x})}{1 - c_1} = \bar{\mathbf{e}} \sin\left(\frac{2\pi x_1}{\ell}\right), \tag{118}$$

where $\bar{\mathbf{e}}$ represents the amplitude tensor of the superimposed deformation and ℓ is its wavelength.

For the sake of simplicity in the numerical examples we assume null pre-existing stress $\mathbf{t}$ within the inclusions,

$$\mathbf{t}_1(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\Sigma}_1(\mathbf{x}) = \mathbf{0}, \tag{119}$$

and we neglect the presence of superimposed infinitesimal rigid-rotations,

$$\begin{aligned}
&\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_2(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{H}_1(\mathbf{x}) = \mathbf{e}_1(\mathbf{x}),\\
&\mathbf{H}_2(\mathbf{x}) = \mathbf{e}_2(\mathbf{x}).
\end{aligned} \tag{120}$$

An example is explored by considering aluminum as the matrix material; in particular we refer to the following Poisson's ratio for the matrix

$$\nu_2 = \frac{3\kappa_2 - 2\mu_2}{2(3\kappa_2 + \mu_2)} = 0.33, \tag{121}$$

and we use the values for the material constants $\beta_2^{(i)}$ reported in [16] and obtained in [17],

$$\begin{aligned}
&\beta_2^{(1)} = 0.89, \qquad \beta_2^{(2)} = 0.96,\\
&\beta_2^{(3)} = -2.63, \qquad \beta_2^{(4)} = -4.54.
\end{aligned} \tag{122}$$

As far as the inclusion phase is concerned, we consider three different cases:

- void inclusions ($\mu_1 = \kappa_1 = 0$);
- rigid inclusions ($\mu_1 = \kappa_1 \to \infty$);
- alumina inclusions ($\mu_1 = 6.65\mu_2$, $\nu_1 = 0.2$).

In the next sections sinusoidal strain fields superimposed either upon a constant prestress for both cases $\mathcal{A}$ and $\mathcal{B}$ or a sinusoidal distribution for the total formulation $\mathcal{A}$ alone are considered.

6.1 Constant prestress

Considering a sinusoidal strain superimposed upon a constant prestress field,

$$\mathbf{t}_2(\mathbf{x}) = \bar{\mathbf{t}}, \tag{123}$$

where $\bar{\mathbf{t}}$ is the 'tensorial' amplitude of the prestress, the non-local description (109) leads to

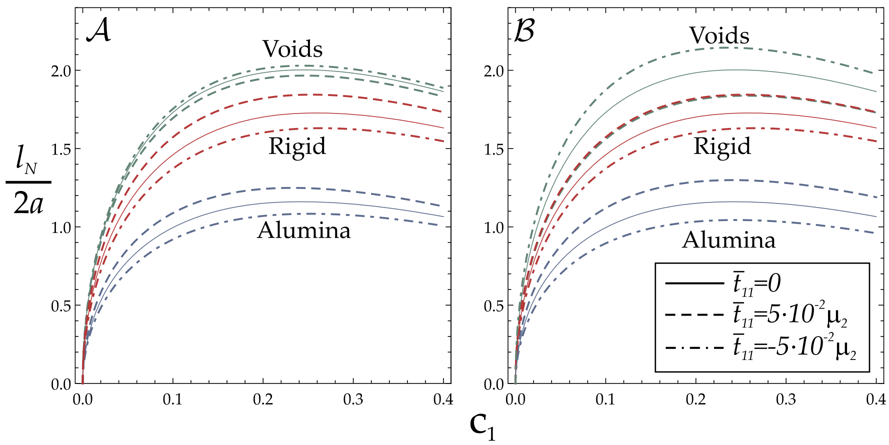

Fig. 1 Dimensionless minimum size of the RVE $\ell_N/2a$ as a function of the concentration of void (*green*), rigid (*red*) and alumina (*blue*, $\nu_1 = 0.2$ and $\mu_1 = 6.65\mu_2$) inclusions c_1 such that an error of 5 %, Eq. (127), is not exceeded with the local effective response given by approaches $\mathcal{A}$ and $\mathcal{B}$, Eqs. (124), (126). Amplitude deformation $\bar{e}_{11}$ is considered superimposed to an amplitude prestress state with only non-null component $\bar{t}_{11} = 5 \cdot \{-10^{-2}; 0; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$ (Color figure online)

$$
\begin{aligned}
&(\mathcal{A})\\
&\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \Bigg\{\Bigg[\mathbb{L}_2 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\Bigg]\\
&\quad \times \left[\bar{\mathbf{e}} - [\mathbb{L}_1 - \mathbb{L}_2]^{-1}\boldsymbol{\Omega}\right]\\
&\quad + \mathbb{L}_1[\mathbb{L}_1 - \mathbb{L}_2]^{-1}\boldsymbol{\Omega}\Bigg\}\sin\left(\frac{2\pi x_1}{\ell}\right)\\
&\quad + \left[\mathbb{L}_1 - \mathbb{L}_2 - \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\right][\mathbb{L}_1 - \mathbb{L}_2]^{-1}\bar{\mathbf{t}},\\
&(\mathcal{B})\\
&\langle\mathring{\boldsymbol{\sigma}}\rangle(\mathbf{x}) = \left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0}) + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}^*\rangle_{,11}(\mathbf{0})\right]\\
&\quad \times \bar{\mathbf{e}}\sin\left(\frac{2\pi x_1}{\ell}\right),
\end{aligned}
\tag{124}
$$

valid for non-rigid inclusions, where

$$
\begin{aligned}
\boldsymbol{\Omega} &= \left(1 + \beta_2^{(4)}\right)(\bar{\mathbf{e}}\bar{\mathbf{t}} + \bar{\mathbf{t}}\bar{\mathbf{e}}) - \mathrm{tr}(\bar{\mathbf{e}})\bar{\mathbf{t}} + \beta_2^{(1)}(\mathrm{tr}\,\bar{\mathbf{e}})(\mathrm{tr}\,\bar{\mathbf{t}})\mathbf{I}\\
&\quad + \beta_2^{(2)}(\mathrm{tr}\,\bar{\mathbf{t}})\bar{\mathbf{e}} + \beta_2^{(3)}\left\{(\mathrm{tr}\,\bar{\mathbf{e}})\bar{\mathbf{t}} + \left[\mathrm{tr}(\bar{\mathbf{e}}\bar{\mathbf{t}})\right]\mathbf{I}\right\},
\end{aligned}
\tag{125}
$$

while in the particular case of rigid inclusions ($\mathbb{L}_1 \to \infty$), the non-local description (109) simplifies as follows:

$$
\begin{aligned}
&(\mathcal{A})\\
&\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \bar{\mathbf{t}} + \Bigg\{(1 - c_1)\Bigg[\mathbb{L}_2 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0})\\
&\quad + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\Bigg]\bar{\mathbf{e}} + \boldsymbol{\Omega}\Bigg\}\sin\left(\frac{2\pi x_1}{\ell}\right),\\
&(\mathcal{B})\\
&\langle\mathring{\boldsymbol{\sigma}}\rangle(\mathbf{x}) = (1 - c_1)\left[\mathbb{L}_0 + \langle\widetilde{\mathbb{T}}^*\rangle(\mathbf{0}) + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}^*\rangle_{,11}(\mathbf{0})\right]\\
&\quad \times \bar{\mathbf{e}}\sin\left(\frac{2\pi x_1}{\ell}\right).
\end{aligned}
\tag{126}
$$

To evaluate the error in the local effective response with respect to the non-local response, we consider the related variations in the only component of stress σ_{ij} (or $\mathring{\sigma}_{ij}$) conjugate to the only non-zero superimposed strain e_{ij}. Since in Eq. (124) we have the sum of a constant value and a sinusoidal function (which amplitude depends both on local and non-local terms), in order to estimate the minimum RVE size, we impose that the ratio between such amplitudes (for the reference stress component σ_{ij} or $\mathring{\sigma}_{ij}$) does not exceed a fixed error-threshold α, i.e.

$$
\left|\frac{\text{Non-local term}_{ij}}{\text{Local term}_{ij}}\right| \leq \alpha. \tag{127}
$$

Criteria (127) yields the estimate of the minimum RVE size ℓ; this is found to behave like the square root of a function of c_1 and of the amplitude $\bar{\mathbf{t}}$.

In Figs. 1 and 2 the influence of uniform normal prestress on the minimum RVE size ℓ_N (superimposed normal average strain, $\bar{e}_{11}$) and ℓ_S (superimposed shear average strain, $\bar{e}_{12}$), respectively, is shown for the three cases considered (voids, rigid inclusions, elastic inclusions formed by alumina).

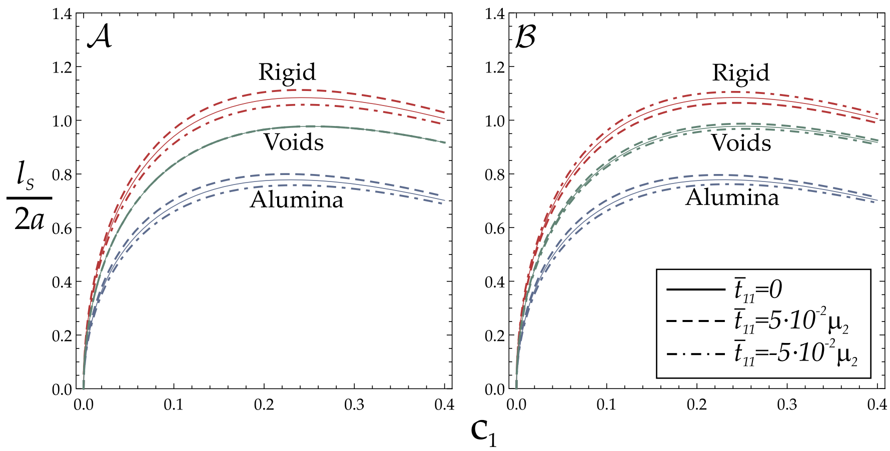

Fig. 2 Dimensionless minimum size of the RVE $\ell_S/2a$ as a function of the concentration of void (*green*), rigid (*red*) and alumina (*blue*, $\nu_1 = 0.2$ and $\mu_1 = 6.65\mu_2$) inclusions c_1 such that an error of 5 %, Eq. (127), is not exceeded with the local effective response given by approach $\mathcal{A}$ and $\mathcal{B}$, Eqs. (124), (126). Amplitude deformation $\bar{e}_{12}$ is considered superimposed to an amplitude prestress state with only non-null component $\bar{t}_{11} = 5 \cdot \{-10^{-2}; 0; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$ (Color figure online)

Figure 1 reveals that both for the total, ($\mathcal{A}$), and for the incremental, ($\mathcal{B}$), approach a normal constant prestress $\bar{t}_{11}$ acting on composites exhibiting randomly distributed spherical rigid inclusions causes high deviation of the RVE size with respect to the un-prestressed case. The total response is also affected by the presence of elastic inclusions (alumina in this case), whereas if voids are present the discrepancy between the prestressed and un-prestressed case are minimal. On the other hand, the incremental response is sensitive to all types of inclusions. The presence of compressive prestress when either elastic or rigid inclusions are present tends to significantly lower the RVE size, whereas tensile prestresses have the opposite effect. This behavior is reversed if voids are present, although this is significant only for the incremental response.

Figure 2 confirms that, similarly to unprestressed materials, the minimum RVE size defined by the superimposition of sinusoidal shear average strain is less restrictive than the corresponding condition arising with the superimposition of sinusoidal normal strain, i.e. $\ell_S < \ell_N$.

Finally, results (not reported here for conciseness) about pure shear prestresses $\bar{t}_{12}$ turn out not to influence the response given by either $\mathcal{A}$ and $\mathcal{B}$ formulations, ultimately leading to the conclusion that constant normal prestresses $\bar{t}_{11}$ are the only responsible for significant deviation of the RVE size from the values achieved for the un-prestressed composite.

6.2 Sinusoidal prestress

In this section formulation $\mathcal{A}$ is employed to study the influence on the RVE size of spatially varying residual stresses. To this purpose, sinusoidal strains superimposed on sinusoidal prestresses are considered, where the latter may be written in the following form:

$$\mathbf{t}_2(\mathbf{x}) = \bar{\mathbf{t}} \sin\left(\frac{2\pi x_1}{L}\right), \tag{128}$$

where the subscript 2 indicates that this is the prestress of the matrix, L is the wavelength characterizing the prestress and $\bar{\mathbf{t}}$ is the 'tensorial' amplitude. By virtue of trigonometric identities, the non-local description (109) leads to the following expression of the total stress

$$\begin{aligned}
&\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) \\
&= \left[\mathbb{L}_2 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\right]\bar{\mathbf{e}} \sin\left(\frac{2\pi x_1}{\ell}\right) \\
&\quad + \left[\mathbb{L}_1 - \mathbb{L}_2 - \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) - \frac{2\pi^2}{L^2}\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\right] \\
&\quad \times [\mathbb{L}_1 - \mathbb{L}_2]^{-1}\bar{\mathbf{t}} \sin\left(\frac{2\pi x_1}{L}\right) \\
&\quad + \frac{1}{2}\left[\mathbb{L}_1 - \mathbb{L}_2 - \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) - 2\pi^2\left(\frac{\ell - L}{\ell L}\right)^2\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\right]
\end{aligned}$$

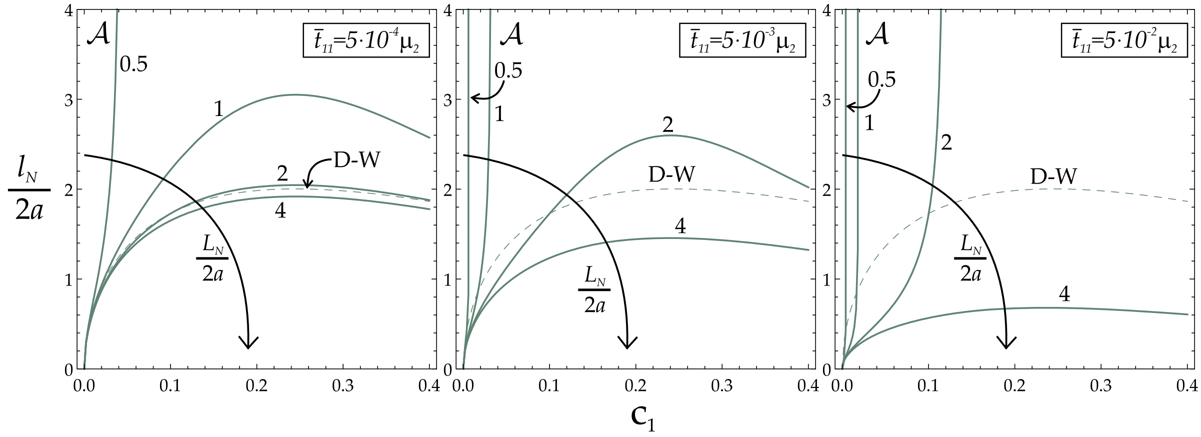

Fig. 3 Dimensionless minimum size of the RVE $\ell_N/2a$ as a function of the concentration of the void inclusions c_1 for different wavelengths of the prestress $L_N = \{0.5; 1; 2; 4\}2a$ such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\overline{e}_{11} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\overline{t}_{11} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

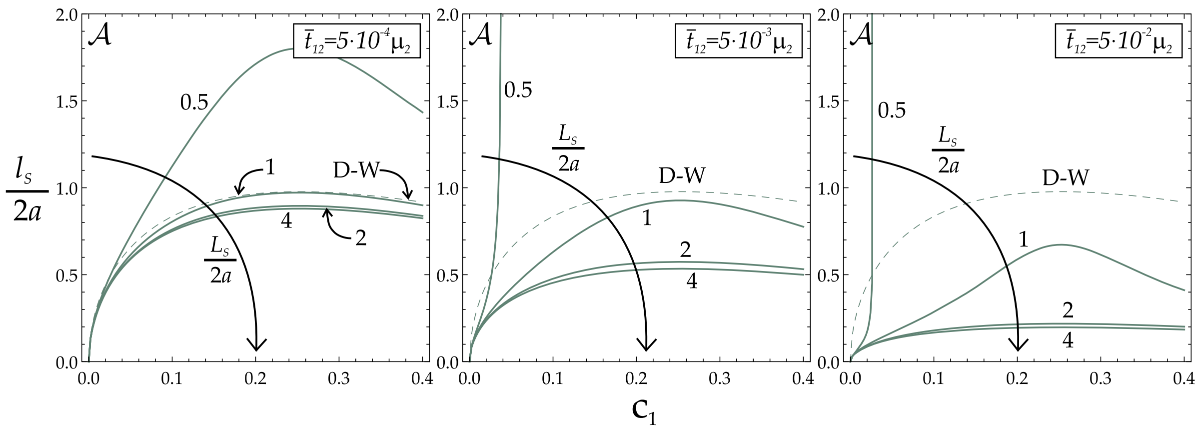

Fig. 4 Dimensionless minimum size of the RVE $\ell_S/2a$ as a function of the concentration of the void inclusions c_1 for different wavelengths of the prestress $L_S = \{0.5; 1; 2; 4\}2a$ such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\overline{e}_{12} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\overline{t}_{12} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

$$\times [\mathbb{L}_1 - \mathbb{L}_2]^{-1} \boldsymbol{\Omega} \cos\left[2\pi x_1 \left(\frac{1}{L} - \frac{1}{\ell}\right)\right]$$
$$-\frac{1}{2}\left[\mathbb{L}_1 - \mathbb{L}_2 - \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) - 2\pi^2\left(\frac{\ell + L}{\ell L}\right)^2 \langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\right]$$
$$\times [\mathbb{L}_1 - \mathbb{L}_2]^{-1} \boldsymbol{\Omega} \cos\left[2\pi x_1 \left(\frac{1}{L} + \frac{1}{\ell}\right)\right], \quad (129)$$

valid for non-rigid inclusions, where $\boldsymbol{\Omega}$ is defined by Eq. (125), and we distinguish four different periodic functions with their respective wavelengths and their local and non-local amplitude.

In the case of rigid inclusion ($\mathbb{L}_1 \to \infty$), the non-local description (109) takes the following simplified form:

($\mathcal{A}$)

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = (1 - c_1)\left[\mathbb{L}_2 + \langle\widetilde{\mathbb{T}}\rangle(\mathbf{0}) + \frac{2\pi^2}{\ell^2}\langle\widetilde{\mathbb{T}}\rangle_{,11}(\mathbf{0})\right]$$

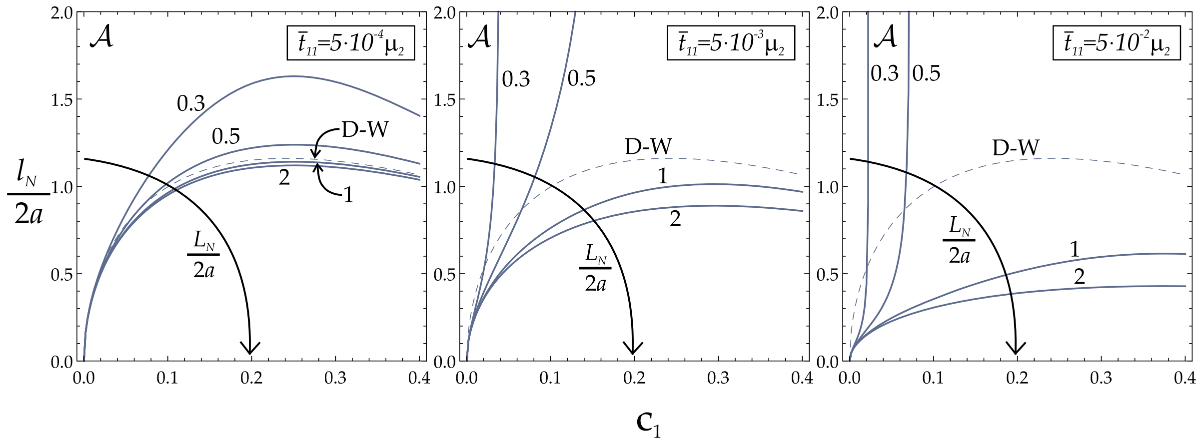

Fig. 5 Dimensionless minimum size of the RVE $\ell_N/2a$ as a function of the concentration of the alumina ($\nu_1 = 0.2$, $\mu_1 = 6.65\mu_2$) inclusions c_1 for different wavelengths of the prestress $L_N = \{0.3; 0.5; 1; 2\}2a$ such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\overline{e}_{11} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\overline{t}_{11} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in an aluminum matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

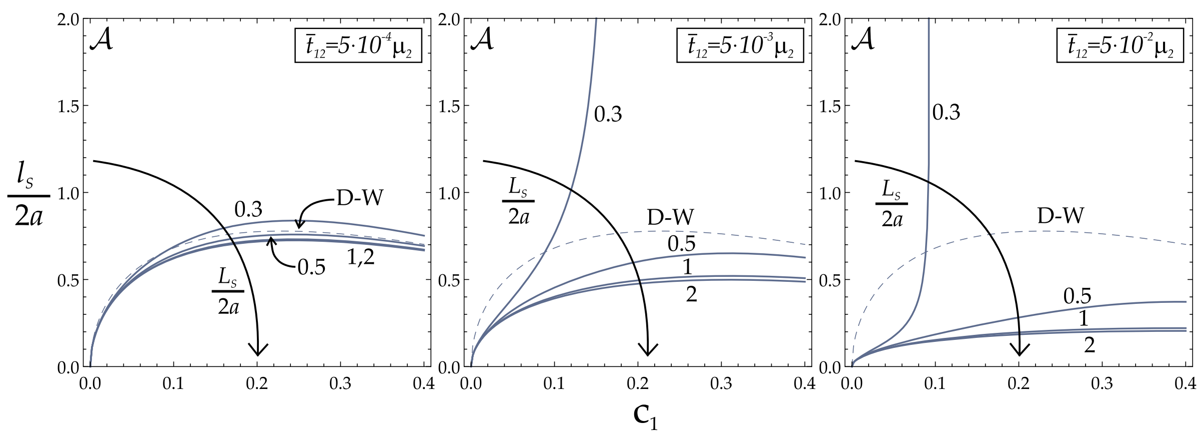

Fig. 6 Dimensionless minimum size of the RVE $\ell_S/2a$ as a function of the concentration of the alumina ($\nu_1 = 0.2$, $\mu_1 = 6.65\mu_2$) inclusions c_1 for different wavelengths of the prestress $L_S = \{0.3; 0.5; 1; 2\}2a$ such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\overline{e}_{12} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\overline{t}_{12} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in an aluminum matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

$$\times \bar{\mathbf{e}} \sin\left(\frac{2\pi x_1}{\ell}\right) + \bar{\mathbf{t}} \sin\left(\frac{2\pi x_1}{L}\right) + \frac{1}{2}\boldsymbol{\Omega}\left\{\cos\left[2\pi x_1\left(\frac{1}{L} - \frac{1}{\ell}\right)\right] - \cos\left[2\pi x_1\left(\frac{1}{L} + \frac{1}{\ell}\right)\right]\right\}. \tag{130}$$

It is worth noting that in the rigid inclusion case the amplitude of the stress depends only on the magnitude of the prestress and not on its wavelength L.

In order to evaluate the error encountered by assuming the local effective constitutive equation instead of the full second gradient non-local response, we compare the corresponding descriptions for the component of stress σ_{ij} conjugate to the only non-null superimposed strain e_{ij}. The minimum RVE size below which the non-local description is required is obtained assuming a 'pointwise' criteria for the maximum value achieved by the reference stress compo-

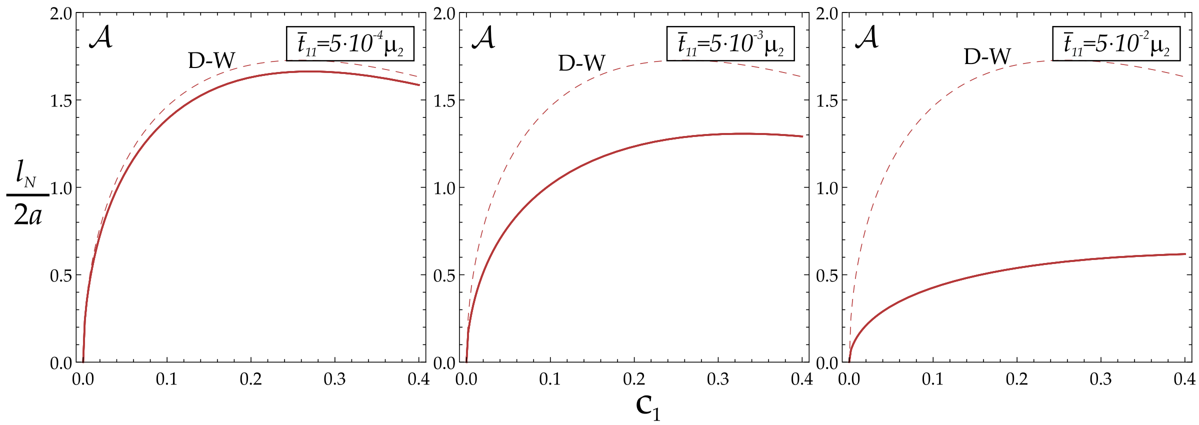

Fig. 7 Dimensionless minimum size of the RVE $\ell_N/2a$ as a function of the concentration of the rigid inclusions c_1 for any value of wavelength of the prestress such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\bar{e}_{11} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\bar{t}_{11} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

nent σ_{ij}, namely

$$\frac{\sum_{c=1}^{4} |\text{Non-local term}_{ij}^{(c)}|}{\sum_{c=1}^{4} |\text{Local term}_{ij}^{(c)}|} \leq \alpha. \tag{131}$$

This enforces that the ratio between the extra-amplitude given by the non-local response and the amplitude given by the local one is bounded by a fixed error α.

The case of matrix with voids is examined at first in Figs. 3 and 4. The former deals with superimposed sinusoidal longitudinal strains, characterized by wavelength ℓ_N, on sinusoidal normal prestresses of wavelength L_N. In this case, no matter what the amplitude of the prestress is, the shorter L_N the higher the amplification on ℓ_N; more specifically, if the ratio between the prestress wave length and the diameter of the voids L_N/a is 0.5 the RVE size measured through the ratio ℓ_N/a tends to be very high even for very dilute composites and, hence, the resulting behavior is fully non-local. This is definitely the case for small and moderate prestress amplitudes (Fig. 3 left and center). In the first of such two cases relatively less oscillating normal prestresses show practically a behavior analogous to the un-prestressed case (labelled with D-W in the figures since the results display the approach of Drugan and Willis in [15]). When higher amplitudes of the prestress are considered (Fig. 3 right), then even oscillations of the prestress relatively moderate, such as $L_N/a = 2$ may cause a rapid increase of the RVE size for relatively diluted composites. From the last two figures it is displayed how a drastic reduction of the RVE may be obtained whenever longitudinal strains are superimposed on relatively slowly oscillating prestresses (such as $L_N/a = 4$) with moderate to high amplitudes.

The analogous of the previous case applied to shear strains superimposed on oscillating shear prestress, Fig. 4. In this situation, unlike for the case of longitudinal strains superimposed on normal prestresses, rapidly oscillating residual shear stresses with small amplitudes do not cause the blow-up of the RVE size (Fig. 4 left). As the amplitude of the prestress increases, this blow-up phenomenon arises even in this case. If the shear prestress has slower oscillations and higher amplitude, the deviation from the un-prestressed case becomes more and more evident, resulting on a drastic reduction on the RVE size required to represent the overall response of the composite with the local term alone for the given threshold α.

Figures 5 and 6 display similar analysis for elastic (alumina) inclusion case, where sinusoidal longitudinal strains and shears are superimposed on sinusoidal normal prestresses respectively. Trends already seen in Fig. 4 for the case of voids are similarly observed for elastic inclusions, even when longitudinal strains act together with normal prestresses. Higher amplitudes and slower spatial oscillation of the prestress significantly reduce the RVE size, whereas the

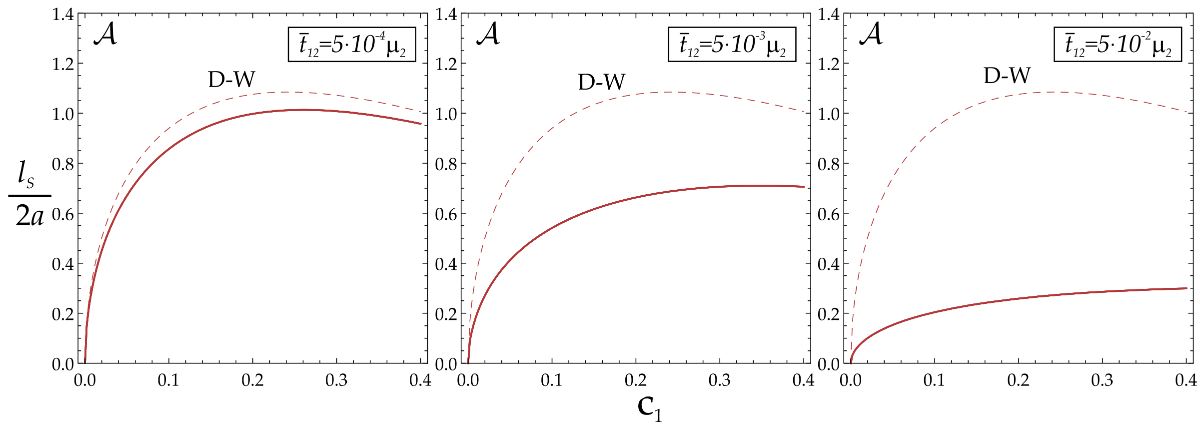

Fig. 8 Dimensionless minimum size of the RVE $\ell_S/2a$ as a function of the concentration of the rigid inclusions c_1 for any value of wavelength of the prestress such that an error of 5 %, Eq. (131), is not exceeded with the local effective response given by approach $\mathcal{A}$, Eq. (129). Amplitude deformation $\overline{e}_{12} = 10^{-3}$ is considered superimposed to an amplitude prestress state with only non-null component $\overline{t}_{12} = 5 \cdot \{10^{-4}; 10^{-3}; 10^{-2}\}\mu_2$ in a matrix with $\nu_2 = 0.33$. Corresponding null prestress (D-W) case (Drugan and Willis [15]) is reported *dashed* (Color figure online)

danger of relatively small amplitudes with very rapid oscillations may result either in a magnification of the RVE size with respect to the un-prestressed case or on its blow-up. The cases of $\overline{e}_{11}$ superimposed to $\overline{t}_{12}$ and $\overline{e}_{12}$ superimposed to $\overline{t}_{11}$ are not displayed in this paper because they give very similar estimates to the RVE size as in the un-prestressed case. Whenever rigid inclusions within an elastic matrix form the random composite the results of superimposing longitudinal strains on normal or shears and shear strains on shear prestresses are displayed in Figs. 7 and 8 respectively. Since in this case no dependence on the prestress wavelength L appears, Eq. (130), no blow-up is ever observed. Furthermore, the RVE size is less than that estimated in [15] without prestress no matter what the concentration of rigid inclusions is. Hence, in such a case, the RVE size estimated without residual stresses is an upper bound on the actual RVE size effectively required to approximate the overall response with the local term alone.

Acknowledgements L. Deseri gratefully acknowledges financial support from grant MIUR-PRIN2008 (2010–2012) "Multiscale modelling, numerical analysis and testing of complex materials and structures for innovative applications", the University of Trento and the Department of Mathematical Sciences and the Center for Nonlinear Analysis through the NSF Grant No. DMS-0635983. L. Deseri and F. Dal Corso gratefully acknowledge financial support from the grant PIAP-GA-2011-286110-INTERCER2, "Modelling and optimal design of ceramic structures with defects and imperfect interfaces". The authors warmly acknowledge Walter J. Drugan for his helpful advice.

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