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# Mindlin second-gradient elastic properties from dilute two-phase Cauchy-elastic composites Part I: Closed form expression for the effective higher-order constitutive tensor

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**Abstract**

It is shown that second-order homogenization of a Cauchy-elastic dilute suspension of randomly distributed inclusions yields an equivalent second gradient (Mindlin) elastic material. This result is valid for both plane and three-dimensional problems and extends earlier findings by Bigoni and Drugan (Analytical derivation of Cosserat moduli via homogenization of heterogeneous elastic materials. *J. Appl. Mech.*, 2007, **74**, 741–753) from several points of view: (i.) the result holds for anisotropic phases with spherical or circular ellipsoid of inertia; (ii.) the displacement boundary conditions considered in the homogenization procedure is independent of the characteristics of the material; (iii.) *a perfect energy match is found* between heterogeneous and equivalent materials (instead of an optimal bound). The constitutive higher-order tensor defining the equivalent Mindlin solid is given in a surprisingly simple formula. Applications, treatment of material symmetries and positive definiteness of the effective higher-order constitutive tensor are deferred to Part II of the present article.

*Keywords*: Second-order homogenization; Higher-order elasticity; Effective non-local continuum; Characteristic length-scale; Composite materials.

## 1 Introduction

Due to the lack of a characteristic length, local constitutive models are unsuitable for mechanical applications at the micro- and nano-scale, since size-effects evidenced by experiments cannot be described and the modelling fails when large strain gradient are present, as in the case of shear band formation (Dal Corso and Willis, 2011). Therefore, many nonlocal models have been formulated and developed, starting from the pioneering work by the Cosserat brothers (1909) and by Koiter (1964) and Mindlin (1964). Despite their evident connection to the microstructure, nonlocal models are usually introduced in a phenomenological way, so that attempts of explicitly relating the microstructure to nonlocal effects are scarce (theoretical considerations were developed by Achenbach and Hermann, 1968; Beran and McCoy, 1970; Boutin, 1996;

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Dal Corso and Deseri, 2013; Forest and Trinh, 2011; Li, 2011; Pideri and Seppecher, 1997; Wang and Stronge, 1999; numerical approaches were given by Auffray et al. 2010; Forest, 1998; Ostoja-Starzewski et al. 1999; Bouyge et al. 2001; experiments were provided by Anderson and Lakes, 1994; Buechner and Lakes, 2003; Lakes, 1986; Gauthier, 1982).

Bigoni and Drugan (2007) have provided a technique to identify Cosserat constants from homogenization of a heterogeneous Cauchy elastic solid. Their approach shows how a nonlocal material can be realized starting from a 'usual' Cauchy elastic composite and opens the way to the practical realization of nonlocal materials. Their methodology has two important limitations, namely, that (i.) the obtained characteristic lengths for the Cosserat material *do not* allow a complete match of the elastic energies between the Cauchy heterogeneous and the Cosserat homogeneous materials, but *minimize* the energy difference between these two, and (ii.) that the homogenization is performed by imposing boundary displacements on the RVE and on the equivalent material depending on the Poisson's ratio of the material (so that the boundary conditions considered are not exactly equal). These two limitations are overcome in the present article, by using a higher-order 'Mindlin' nonlocal elastic material which provides a perfect match between the elastic energies of a dilute suspension of Cauchy-elastic inclusions (randomly distributed in a Cauchy-elastic matrix) and a homogeneous non-local elastic material, obtained through application of the same displacement field at the boundary. Moreover, although our results remain confined to the dilute assumption, we also generalize Bigoni and Drugan (2007) by relaxing (iii.) the restriction of isotropy and (iv.) the shape of the inclusions, which may now have a generic form (though subject to certain geometrical restrictions to be detailed later).

Description of the proposed identification procedure of the Mindlin elastic constants and the relative closed-form formulae are reported in this article, while a discussion about positive-definiteness, material symmetries and applications to explicit cases are deferred to Part II.

## 2 Preliminaries on Second-Gradient Elasticity (SGE)

The equations are briefly introduced governing the equilibrium of the second-gradient elastic (SGE) solid proposed by Mindlin and Eshel (1968)[1] that will be employed in the homogenization procedure.

Considering a quasi-static deformation process, defined by the displacement field $\boldsymbol{u}$ (function of the position $\boldsymbol{x}$), the primary kinematical quantities of the SGE are defined as

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \qquad \chi_{ijk} = u_{k,ij}, \tag{1}$$

where a comma denotes differentiation, the indices range between 1 and $N$ (equal to 2 or 3, depending on the space dimensions of the problem considered), and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ are the (second-order) strain and the (third-order) curvature tensor fields, respectively, satisfying the following symmetry properties

$$\varepsilon_{ij} = \varepsilon_{ji}, \qquad \chi_{ijk} = \chi_{jik}. \tag{2}$$

Defining the statical entities Cauchy stress $\sigma_{ij}{=}\sigma_{ji}$ and double stress $\tau_{ijk}{=}\tau_{jik}$, respectively work-conjugate to the kinematical entities $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$, eqn (1), the principle of virtual work can be written for a solid occupying a domain $\Omega$, with boundary $\partial\Omega$ and set of edges $\Gamma$, in the

[1]Note that the linear elastic second-gradient (of the displacement) model is fully equivalent to the linear elastic first-gradient (of the strain) model (Mindlin and Eshel; 1968).

absence of body-force as

$$\int_\Omega (\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\chi_{ijk}) = \int_{\partial\Omega} (t_i\delta u_i + T_i D\delta u_i) + \int_\Gamma \Theta_i \delta u_i, \tag{3}$$

where repeated indices are summed, $\boldsymbol{t}$ represents the surface traction (work-conjugate to $\boldsymbol{u}$), while $\boldsymbol{T}$ and $\boldsymbol{\Theta}$ denote the generalized tractions on the surface $\partial\Omega$ and along the set of edges $\Gamma$ (work-conjugate respectively to $D\boldsymbol{u}$ and $\boldsymbol{u}$), and $D = n_l\partial_l$ represents the derivative along the outward normal direction to the boundary $\boldsymbol{n}$ (only on $\partial\Omega$ but not on $\Gamma$). Through integration by parts, the equilibrium conditions, holding for points within the body $\Omega$, can be obtained as

$$\partial_j\left(\sigma_{jk} - \partial_i\tau_{ijk}\right) = 0, \qquad \text{in } \Omega, \tag{4}$$

while for points on the boundary $\partial\Omega_p$ and along the set of edges $\Gamma_p$, (where statical conditions are prescribed in terms of $\boldsymbol{t}$, $\boldsymbol{T}$ and $\boldsymbol{\Theta}$) as

$$\begin{cases} n_j\sigma_{jk} - n_in_jD\tau_{ijk} - 2n_jD_i\tau_{ijk} + \left(n_in_jD_ln_l - D_jn_i\right)\tau_{ijk} = t_k, \\ \\ n_in_j\tau_{ijk} = T_k, \end{cases} \qquad \text{on } \partial\Omega_p, \tag{5}$$

and

$$[\![ e_{mlj}n_is_mn_l\tau_{ijk} ]\!] = \Theta_k, \qquad \text{on } \Gamma_p, \tag{6}$$

where $e_{mlj}$ is the Ricci 'permutation' tensor, $D_j = \left(\delta_{jl} - n_jn_l\right)\partial_l$, $\boldsymbol{s}$ is the unit vector tangent to $\Gamma$ and $[\![\cdot]\!]$ represents the jump of the enclosed quantity, computed with the normals $\boldsymbol{n}$ defined on the surfaces intersecting at the edge $\Gamma$. Finally, kinematical conditions[2] are prescribed for points on the remaining boundary $\partial\Omega_u \equiv \partial\Omega\backslash\partial\Omega_p$ as

$$\begin{cases} u_i = \overline{u}_i, \\ \\ Du_i = \overline{Du}_i, \end{cases} \qquad \text{on } \partial\Omega_u. \tag{7}$$

Introducing the strain energy density $w^{SGE} = w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$, the $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ fields can be obtained as

$$\sigma_{ij} = \frac{\partial w^{SGE}}{\partial\varepsilon_{ij}}, \qquad \tau_{ijk} = \frac{\partial w^{SGE}}{\partial\chi_{ijk}}, \tag{8}$$

so that, restricting attention to centrosymmetric materials within a linear theory[3], it follows that

$$w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \underbrace{\frac{1}{2}\mathsf{C}_{ijhk}\varepsilon_{ij}\varepsilon_{hk}}_{w^{SGE,L}(\boldsymbol{\varepsilon})} + \underbrace{\frac{1}{2}\mathsf{A}_{ijklmn}\chi_{ijk}\chi_{lmn}}_{w^{SGE,NL}(\boldsymbol{\chi})}, \tag{9}$$

where $\mathsf{C}$ and $\mathsf{A}$ are the local (fourth-order) and non-local (sixth-order) constitutive tensors, each generating respectively a strain energy density contribution, say 'local', $w^{SGE,L}$ (corresponding to the energy stored in a Cauchy material, $w^{SGE,L} = w^C$) and 'non-local', $w^{SGE,NL}$. Therefore, the linear constitutive equations for the stress and double stress quantities are obtained as

$$\sigma_{ij} = \mathsf{C}_{ijhk}\varepsilon_{hk}, \qquad \tau_{ijk} = \mathsf{A}_{ijklmn}\chi_{lmn}, \tag{10}$$

[2]In the proposed homogenization procedure only kinematical boundary conditions will be imposed ($\partial\Omega_p \equiv \emptyset$, so that $\partial\Omega_u \equiv \partial\Omega$).

[3]Centrosymmetry is coherent with the fact that the elastic energies at first- and at second- order are decoupled under the geometrical assumptions that will be introduced in Section 3.1.

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which, from eqns (1) and (8), have the following symmetries

$$\mathbf{C}_{ijhk} = \mathbf{C}_{jihk} = \mathbf{C}_{ijkh} = \mathbf{C}_{hkij}, \qquad \mathbf{A}_{ijklmn} = \mathbf{A}_{jiklmn} = \mathbf{A}_{ijkmln} = \mathbf{A}_{lmnijk}. \tag{11}$$

In the case of isotropic response, the constitutive elastic tensors $\mathbf{C}$ and $\mathbf{A}$ can be written in the following form

$$\begin{aligned}
\mathbf{C}_{ijhk} = {} & \lambda\delta_{ij}\delta_{hk} + \mu(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \\
\mathbf{A}_{ijhlmn} = {} & \frac{a_1}{2}\left[\delta_{ij}\left(\delta_{hl}\delta_{mn} + \delta_{hm}\delta_{ln}\right) + \delta_{lm}\left(\delta_{in}\delta_{jh} + \delta_{ih}\delta_{jn}\right)\right] \\
& + \frac{a_2}{2}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln}\right) + \delta_{jh}\left(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln}\right)\right] \\
& + 2\,a_3\left(\delta_{ij}\delta_{hn}\delta_{lm}\right) + a_4\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{hn} \\
& + \frac{a_5}{2}\left[\delta_{in}\left(\delta_{jl}\delta_{hm} + \delta_{jm}\delta_{hl}\right) + \delta_{jn}\left(\delta_{il}\delta_{hm} + \delta_{im}\delta_{hl}\right)\right],
\end{aligned} \tag{12}$$

where $\delta_{ij}$ is the Kronecker delta, $\lambda$ and $\mu$ are the usual Lamé constants, defining the local isotropic behavior, while $a_i$ $(i = 1, ..., 5)$ are the five material constants (with the dimension of a force) defining the nonlocal isotropic behavior. Considering the constitutive isotropic tensors (12), the strain energy density (9) becomes

$$w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \underbrace{\frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij}}_{w^{SGE,L}(\boldsymbol{\varepsilon})} + \underbrace{\sum_{k=1}^{5} a_k \mathcal{I}_k(\boldsymbol{\chi})}_{w^{SGE,NL}(\boldsymbol{\chi})}, \tag{13}$$

where the invariants $\mathcal{I}_k(\boldsymbol{\chi})$ are

$$\begin{aligned}
&\mathcal{I}_1(\boldsymbol{\chi}) = \chi_{iik}\,\chi_{jkj}(= \chi_{iik}\,\chi_{kjj}), \\
&\mathcal{I}_2(\boldsymbol{\chi}) = \chi_{iki}\,\chi_{jkj}(= \chi_{kii}\,\chi_{jkj} = \chi_{kii}\,\chi_{kjj} = \chi_{iki}\,\chi_{kjj}), \\
&\mathcal{I}_3(\boldsymbol{\chi}) = \chi_{iik}\,\chi_{jjk}, \\
&\mathcal{I}_4(\boldsymbol{\chi}) = \chi_{ijk}\,\chi_{ijk}(= \chi_{jik}\,\chi_{ijk} = \chi_{jik}\,\chi_{jik} = \chi_{ijk}\,\chi_{jik}), \\
&\mathcal{I}_5(\boldsymbol{\chi}) = \chi_{ijk}\,\chi_{kji}(= \chi_{jik}\,\chi_{kji} = \chi_{jik}\,\chi_{jki} = \chi_{ijk}\,\chi_{jki}),
\end{aligned} \tag{14}$$

so that the linear constitutive relations (10) reduce to

$$\begin{aligned}
\sigma_{ij} = {} & \lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij}, \\
\tau_{ijk} = {} & \frac{a_1}{2}\left(\chi_{lli}\delta_{jk} + 2\chi_{kll}\delta_{ij} + \chi_{llj}\delta_{ik}\right) + a_2\left(\chi_{ill}\delta_{jk} + \chi_{jll}\delta_{ik}\right) + 2a_3\chi_{llk}\delta_{ij} \\
& + 2a_4\chi_{ijk} + a_5\left(\chi_{kji} + \chi_{kij}\right).
\end{aligned} \tag{15}$$

Since the invariants defined by eqns (14) satisfy the following inequalities

$$\begin{gathered}
2\mathcal{I}_1(\boldsymbol{\chi}) + \mathcal{I}_2(\boldsymbol{\chi}) + \mathcal{I}_3(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_2(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_3(\boldsymbol{\chi}) \geq 0, \\
\mathcal{I}_4(\boldsymbol{\chi}) \geq 0, \qquad \mathcal{I}_4(\boldsymbol{\chi}) + \mathcal{I}_5(\boldsymbol{\chi}) \geq 0,
\end{gathered} \tag{16}$$

the positive definiteness condition for the isotropic strain energy density $w^{SGE}(\boldsymbol{\varepsilon}, \boldsymbol{\chi})$, eqn (13), corresponds to the usual restraints for the local parameters (given by the positive definiteness of $w^{SGE,L}(\boldsymbol{\varepsilon})$)

$$3\lambda + 2\mu > 0, \quad \mu > 0, \tag{17}$$

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which are complemented by the following conditions (Mindlin and Eshel, 1968) on the nonlocal constitutive parameters (given by the positive definiteness of $w^{SGE,NL}(\boldsymbol{\chi})$)

$$-a_4 < a_5 < 2a_4, \quad e_1 > 0, \quad e_2 > 0, \quad 5e_3^2 < 2e_1e_2, \tag{18}$$

where

$$\begin{gathered} e_1 = -4a_1 + 2a_2 + 8a_3 + 6a_4 - 3a_5, \quad e_2 = 5(a_1 + a_2 + a_3) + 3(a_4 + a_5), \\ e_3 = a_1 - 2a_2 + 4a_3. \end{gathered} \tag{19}$$

## 3 Homogenization procedure

The proposed homogenization procedure follows Bigoni and Drugan (2007). In particular, the same[4] (linear and quadratic) displacement is applied on the boundary of both the representative volume element RVE and the homogeneous equivalent SGE material. Then, the equivalent local $\mathbf{C}^{eq}$ and non-local $\mathbf{A}^{eq}$ tensors are obtained imposing the vanishing of the elastic energy mismatch between the two materials. Since the strain energy in the homogeneous SGE material is given only by the local contribution when linear displacement boundary condition are applied (because no strain gradient arises), the equivalent local tensor $\mathbf{C}^{eq}$ corresponds to that obtained with usual homogenization procedures. Thus, the remaining unknown of the equivalent SGE material (namely, the non-local equivalent constitutive tensor $\mathbf{A}^{eq}$) can be obtained by imposing the vanishing mismatch in strain energy when (linear and) quadratic displacement are considered. A chief result in the current procedure is that a perfect match in the elastic energies is achieved, while Bigoni and Drugan (2007) only obtained an ‘optimality condition’ for the mismatch.

The homogenization procedure is described in the following three steps, where reference is made to a generic RVE, although results will be presented for a diluted distribution of randomly located inclusions.

**Step 1.** Consider a RVE made up of a heterogeneous Cauchy material (C), Fig. 1 (left), occupying a region

$$\Omega^C_{RVE} \equiv \Omega^C_1 \cup \Omega^C_2,$$

where an inclusion, phase ‘2’ (occupying the region $\Omega^C_2$ and with elastic tensor $\mathbf{C}^{(2)}$), is fully enclosed in a matrix, phase ‘1’ (occupying the region $\Omega^C_1$ and with elastic tensor $\mathbf{C}^{(1)}$), so that the constitutive local tensor $\mathbf{C}(\boldsymbol{x})$ within the RVE can be defined as the piecewise constant function

$$\mathbf{C}(\boldsymbol{x}) = \begin{cases} \mathbf{C}^{(1)} & \boldsymbol{x} \in \Omega^C_1, \\ \\ \mathbf{C}^{(2)} & \boldsymbol{x} \in \Omega^C_2, \end{cases} \tag{20}$$

and the volume fraction $f$ of the inclusion phase can be defined as

$$f = \frac{\Omega^C_2}{\Omega^C_{RVE}}. \tag{21}$$

[4]Bigoni and Drugan (2007) impose a linear and quadratic displacement field on the boundaries of the RVE and of the homogeneous equivalent material, whose quadratic part depends on the Poisson’s ratio of the material to which the displacement is applied, so that the applied displacements are not exactly equal. Furthermore, the equivalent material considered by Bigoni and Drugan is a non-local Koiter material (1964), which does not permit the annihilation, but only a minimization of the elastic energy mismatch between the RVE and the equivalent material.

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The equivalent material is a homogeneous SGE material, Fig. 1 (right), occupying the region $\Omega_{eq}^{SGE}$

$$\Omega_{eq}^{SGE} = \Omega_{RVE}^{C}, \tag{22}$$

and constitutive elastic tensors $\mathbf{C}^{eq}$ (local part) and $\mathbf{A}^{eq}$ (nonlocal part). Since the region $\Omega_{eq}^{SGE}$ of the equivalent SGE material corresponds by definition to the region $\Omega_{RVE}^{C}$ of the heterogeneous RVE, in the following both these domains may be identified as $\Omega$.


Figure 1: Left: Heterogeneous Cauchy-elastic RVE where a matrix of elastic tensor $\mathbf{C}^{(1)}$ contains a generic inclusion of elastic tensor $\mathbf{C}^{(2)}$. Right: Homogeneous equivalent SGE material with local tensor $\mathbf{C}^{eq}$ and nonlocal tensor $\mathbf{A}^{eq}$.

**Step 2.** Impose on the RVE boundary the following second-order (linear and quadratic) displacement field $\overline{\boldsymbol{u}}$, Fig. 2 (left)

$$\boldsymbol{u} = \overline{\boldsymbol{u}}, \quad \text{on } \partial\Omega_{RVE}^{C}, \tag{23}$$

with

$$\overline{u}_i = \underbrace{\alpha_{ij} x_j}_{\overline{u}_i^{\alpha}} + \underbrace{\beta_{ijk} x_j x_k}_{\overline{u}_i^{\beta}}, \tag{24}$$

where $\alpha_{ij}$ and $\beta_{ijk}$ are constant coefficients, the latter having the symmetry $\beta_{ijk}=\beta_{ikj}$.

Impose on the equivalent homogeneous SGE boundary again the displacement (24), but together with its normal derivative, Fig. 2 (right), so that

$$\begin{cases} \boldsymbol{u} = \overline{\boldsymbol{u}}, \\ D\boldsymbol{u} = D\overline{\boldsymbol{u}}, \end{cases} \quad \text{on } \partial\Omega_{eq}^{SGE}. \tag{25}$$

Note that the mean value of the local strain gradient, which cannot be controlled solely by Dirichlet conditions, is defined by *imposing* the Neumann condition $(25)_2$. This condition can be justified through consideration of the dilute assumption, so that the influence of the inclusion on the normal derivative is negligible near the boundary of the RVE.

The imposition of the boundary conditions (23) on the RVE and (25) on the equivalent SGE corresponds, respectively, to the two strain energies

$$\mathcal{W}_{RVE}^{C} = \int_{\Omega_1^C} w^C\big|_{\mathbf{C}^{(1)}} + \int_{\Omega_2^C} w^C\big|_{\mathbf{C}^{(2)}}, \qquad \mathcal{W}_{eq}^{SGE} = \int_{\Omega_{eq}^{SGE}} w^{SGE}\big|_{\mathbf{C}^{eq},\mathbf{A}^{eq}}, \tag{26}$$

so that for a generic quadratic displacement field, eqn. (24), an energy mismatch (or 'gap') $\mathcal{G}$ between the two materials arises as a function of the unknown equivalent constitutive tensor $\mathbf{A}^{eq}$

$$\mathcal{G}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = \mathcal{W}_{RVE}^{C} - \mathcal{W}_{eq}^{SGE}. \tag{27}$$

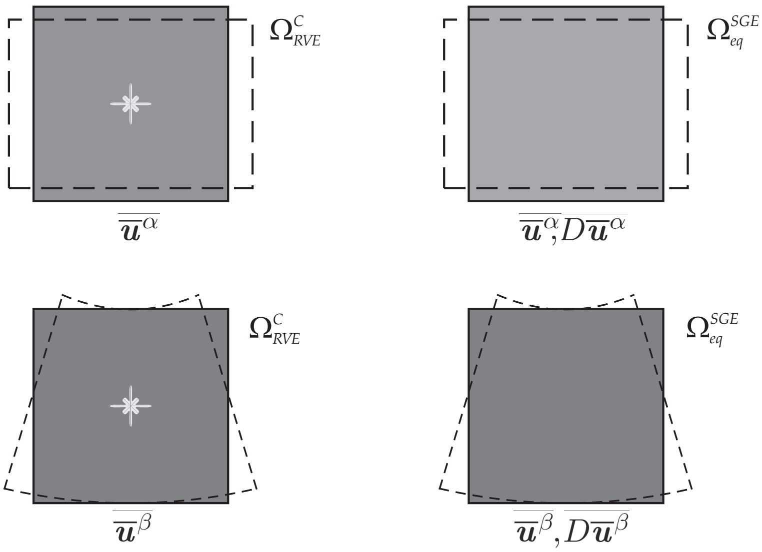


Figure 2: Imposition of the same linear (top) and quadratic (bottom) boundary displacement conditions on the heterogeneous Cauchy RVE (left) and on the homogeneous equivalent SGE (right). In the homogeneous equivalent SGE (right) the normal derivative of displacement (Neumann condition) is also imposed at the boundary.

**Step 3.** Find the unknown equivalent constitutive tensor $\mathsf{A}^{eq}$ by imposing a null energy mismatch $\mathcal{G}$

$$\mathcal{G}\left(\mathsf{C}^{(1)}, \mathsf{C}^{(2)}, \mathsf{C}^{eq}, \mathsf{A}^{eq}\right) = 0. \tag{28}$$

Note that in the case of purely linear displacements ($\boldsymbol{\beta} = \mathbf{0}$) the energy mismatch $\mathcal{G}$ is null by definition of $\mathsf{C}^{eq}$. On the other hand, when quadratic displacements are considered, an energy mismatch $\mathcal{G}$ is different from zero and can be tuned to vanish by changing the value of the unknown tensor $\mathsf{A}^{eq}$.

The above-procedure is general, but subsequent calculations will be limited to the dilute approximation, and the results will be an extension of Bigoni and Drugan (2007) since (i.) the inclusions are of arbitrary shape and, more interestingly, (ii.) the comparison material, a Mindlin elastic second-gradient material, allows a perfect match of the energies (while Bigoni and Drugan (2007) did consider only cylindrical or spherical inclusions and were only able to provide a minimization of the energy gap).

## 3.1 Assumptions about geometrical properties of matrix and inclusion phases

Henceforth the following geometrical properties for both the subsets $\Omega^C_1$and $\Omega^C_2$ will be assumed:[5]

[5]Note that, by definition of static moment vector $\boldsymbol{S}$ and Euler tensor of inertia $\boldsymbol{E}$, eqn (33), the geometrical properties **GP1**, eqn (31) and **GP2**, eqn (32), of the subsets $\Omega^C_1$and $\Omega^C_2$ are also necessarily satisfied by $\Omega^C_{RVE}$, so that

$$\boldsymbol{S}(\Omega^C_{RVE}) = \mathbf{0}, \qquad \boldsymbol{E}(\Omega^C_{RVE}) = \rho^2 \Omega^C_{RVE} \boldsymbol{I}, \tag{29}$$

where the radius $\rho = \rho(\Omega^C_{RVE})$ is related to the radii of the matrix $\rho^{(1)}$ and the inclusion $\rho^{(2)}$ as follows

$$\rho^2 = (1 - f)\left[\rho^{(1)}\right]^2 + f\left[\rho^{(2)}\right]^2. \tag{30}$$

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**GP1)** The centroids of the matrix and of the inclusion coincide and correspond to the origin of the $x_i$–axes, so that both the static moments of the inclusion and of the matrix are null

$$\boldsymbol{S}(\Omega_1^C) = \boldsymbol{0}, \qquad \boldsymbol{S}(\Omega_2^C) = \boldsymbol{0}. \tag{31}$$

**GP2)** The $x_i$–axes are principal axes of inertia for both the matrix and the inclusion and the ellipsoids of inertia are a sphere (or a circle in 2D)

$$\boldsymbol{E}(\Omega_1^C) = \left[\rho^{(1)}\right]^2 \Omega_1^C \boldsymbol{I}, \qquad \boldsymbol{E}(\Omega_2^C) = \left[\rho^{(2)}\right]^2 \Omega_2^C \boldsymbol{I}, \tag{32}$$

where $\boldsymbol{I}$ is the identity second-order tensor and the second-order Euler tensor of inertia $\boldsymbol{E}$ relative to the $x_i$–axes, defined for a generic solid occupying the region $V$ as

$$E_{ij}(V) = \int_V x_i\, x_j, \tag{33}$$

while $\rho^{(1)} = \rho(\Omega_1^C)$ and $\rho^{(2)} = \rho(\Omega_2^C)$ are the radii of the spheres (or circles in 2D) of inertia of the matrix and the inclusion. Note that the assumption of spherical tensors of inertia yields a spherical tensor for the RVE, which is coherent with the assumption of randomness of the distribution of inclusions.

**GP3)** The radius of the sphere of inertia for the inclusion phase vanishes in the limit of null inclusion volume fraction

$$\lim_{f\to 0} \rho^{(2)}(f) = 0, \tag{34}$$

or, equivalently, all the dimensions of the inclusion (and therefore the radius of the smallest ball containing the inclusion) are zero for $f = 0$.

Examples of two-dimensional RVE, characterized by the geometrical properties **GP1-GP2** and **GP3** are reported in Figs. 3 and 4, respectively.

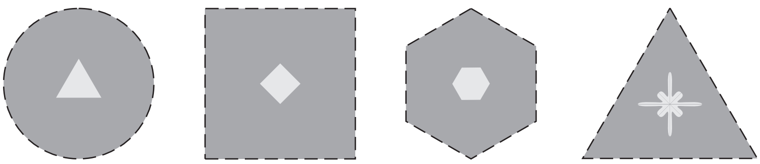

Figure 3: Some examples of two-dimensional RVE satisfying the geometrical properties **GP1**, eqn (31), and **GP2**, eqn (32), for plane strain condition.

## 4 Equivalent nonlocal properties from homogenization in the dilute case

The following proposition is the central result in this article, providing *the nonlocal effective tensor from second-order homogenization of a heterogeneous Cauchy RVE containing a small inclusion.*

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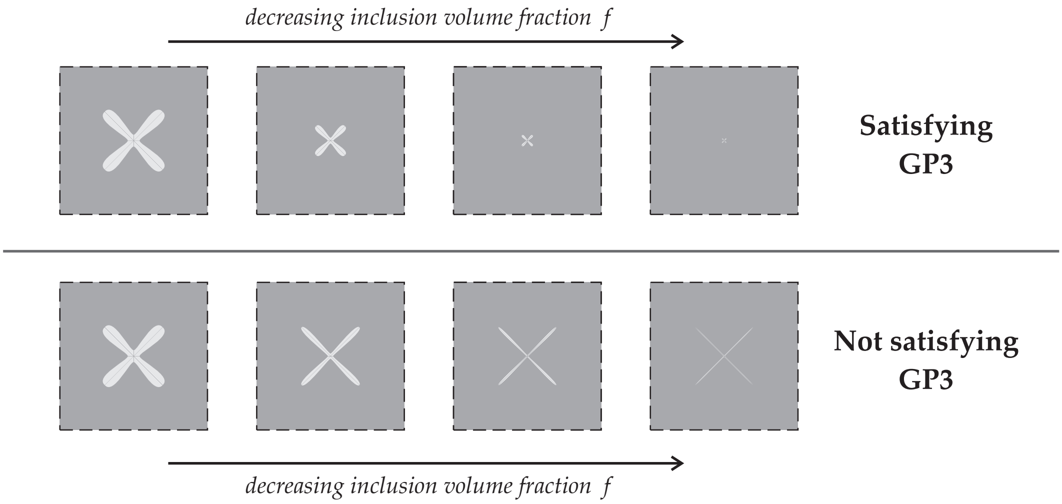


Figure 4: Examples of two-dimensional RVE satisfying (upper part) or not (lower part) the geometrical property **GP3**, eqn (34). In the lower part, the radius of inertia of the inclusion does not vanish in the limit of vanishing volume fraction.

**Homogenization proposition.** For a dilute concentration of the inclusion phase ($f \ll 1$) and assuming the geometrical properties **GP1 - GP2 - GP3** for the RVE, the nonlocal sixth-order tensor $\mathbf{A}^{eq}$ of the equivalent SGE material is evaluated (at first-order in $f$) as

$$\mathbf{A}^{eq}_{ijhlmn} = -f\frac{\rho^2}{4}\left(\tilde{\mathbf{C}}_{ihln}\delta_{jm} + \tilde{\mathbf{C}}_{ihmn}\delta_{jl} + \tilde{\mathbf{C}}_{jhln}\delta_{im} + \tilde{\mathbf{C}}_{jhmn}\delta_{il}\right) + o(f), \tag{35}$$

where $\rho$ is the radius of the sphere (or circle in 2D) of inertia of the RVE cell, and $\tilde{\mathbf{C}}$ is introduced to define (at first-order in $f$) the difference between the local constitutive tensors for the effective material $\mathbf{C}^{eq}$ and the matrix $\mathbf{C}^{(1)}$, so that

$$\mathbf{C}^{eq} = \mathbf{C}^{(1)} + f\tilde{\mathbf{C}}, \tag{36}$$

which is assumed to be known from standard homogenization, performed on linear displacement boundary conditions.

Eqn (35) represents the solution of the homogenization problem and is obtained by imposing the vanishing of the energy mismatch $\mathcal{G}$, eqn (28), when the same second-order displacement boundary conditions are applied both on the heterogeneous Cauchy material and on the homogeneous equivalent SGE material, eqns (23) and (25), respectively.

From the solution (35), in agreement with Bigoni and Drugan (2007), it can be noted that:

- the equivalent SGE material is positive definite if and only if $\tilde{\mathbf{C}}$ is negative definite;
- the constitutive higher-order tensor $\mathbf{A}^{eq}$ is linear in $f$ for dilute concentration.

**Proof of the homogenization proposition**

i) Consider the second-order (linear and quadratic) displacement boundary condition (25) applied on the boundary of a homogeneous SGE material with constitutive tensors $\mathbf{C}$ and

**A**. In the absence of body force, $\boldsymbol{b} = \mathbf{0}$, let us consider the extension within the body of the quadratic displacement field $\overline{\boldsymbol{u}}$, eqn (24), applied on the boundary

$$u_i = \underbrace{\alpha_{ij} x_j}_{u_i^\alpha} + \underbrace{\beta_{ijk} x_j x_k}_{u_i^\beta}, \qquad \boldsymbol{x} \text{ in } \Omega, \tag{37}$$

providing the following deformation $\boldsymbol{\varepsilon}$ and curvature $\boldsymbol{\chi}$ fields

$$\varepsilon_{ij} = \frac{\alpha_{ij} + \alpha_{ji}}{2} + (\beta_{ijk} + \beta_{jik}) x_k, \qquad \chi_{ijk} = 2\beta_{kij}, \tag{38}$$

and the following stress $\boldsymbol{\sigma}$ and double-stress $\boldsymbol{\tau}$ fields,

$$\sigma_{ij} = \mathsf{C}_{ijhk} \alpha_{hk} + 2\mathsf{C}_{ijhk} \beta_{hkl} x_l, \qquad \tau_{ijk} = 2\mathsf{A}_{ijklmn} \beta_{nlm}. \tag{39}$$

The stress field (39) follows from the displacement field (37) and satisfies the equilibrium equation (4) if and only if [6]

$$\mathsf{C}_{ijhk} \beta_{hkj} = \mathbf{0}, \tag{40}$$

which for isotropic homogeneous materials reduces to the condition obtained by Bigoni and Drugan (2007)

$$\beta_{jji} = -(1 - 2\nu) \beta_{ikk}, \tag{41}$$

(with Poisson's ratio $\nu = \lambda / 2(\lambda + \mu)$).

In the following we will use the superscript $^\diamond$ for $\boldsymbol{\beta}$ (namely, $\boldsymbol{\beta}^\diamond$) to denote the components of the third-order tensor $\boldsymbol{\beta}$ satisfying eqn (40), or (41) for isotropy.

ii) Consider an auxiliary material with local constitutive tensor $\mathsf{C}^*$, defined as a first-order perturbation in $f$ to the equivalent local constitutive tensor $\mathsf{C}^{eq}$, namely,

$$\mathsf{C}^* = \mathsf{C}^{eq} + f \left( \hat{\mathsf{C}} - \tilde{\mathsf{C}} \right), \tag{42}$$

so that using eqn (36) we can write

$$\mathsf{C}^* = \mathsf{C}^{(1)} + f \hat{\mathsf{C}}, \tag{43}$$

where $\hat{\mathsf{C}}$, together with $\mathsf{C}^*$, define an arbitrary material with properties 'close' to both the matrix and the equivalent material, an arbitrariness which will be used later to eliminate the constraint (40). By definition, the displacement field

$$u_i^* = \underbrace{\alpha_{ij} x_j}_{u_i^\alpha} + \underbrace{\beta_{ijk}^{\diamond *} x_j x_k}_{u_i^{\beta \diamond *}}, \qquad \boldsymbol{x} \text{ in } \Omega. \tag{44}$$

is equilibrated [in other words satisfies eqn (40)] in a homogeneous material characterized by the constitutive tensor $\mathsf{C}^*$ and corresponds to the following quadratic displacement field on the boundary

$$\overline{u}_i^* = \underbrace{\alpha_{ij} x_j}_{\overline{u}_i^\alpha} + \underbrace{\beta_{ijk}^{\diamond *} x_j x_k}_{\overline{u}_i^{\beta \diamond *}}, \qquad \boldsymbol{x} \text{ on } \partial\Omega. \tag{45}$$

[6] Note that the constraint (40) arises independently of whether the material is Cauchy elastic or SGE.

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iii) Apply on the boundary $\partial\Omega^C_{RVE}$ of the heterogeneous Cauchy material (RVE) the displacement boundary condition (45),

$$\overline{\boldsymbol{u}}^{RVE} = \overline{\boldsymbol{u}}^*, \qquad \text{on } \partial\Omega^C_{RVE}. \tag{46}$$

According to **Lemma 1** (Appendix A.1), the strain energy in the RVE at first-order in $f$ is the sum of the strain energy due to the linear ($\boldsymbol{\alpha}$) and nonlinear ($\boldsymbol{\beta}$) displacement boundary conditions, and the mutual strain energy, say, the '$\boldsymbol{\alpha}-\boldsymbol{\beta}$ energy term' is null at first-order in $f$,[7] so that

$$\mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^{\beta\diamond *}\right) + o(f). \tag{48}$$

iv) Apply on the boundary $\partial\Omega^{SGE}_{eq}$ of the homogeneous SGE material the same displacement boundary condition $\overline{\boldsymbol{u}}^*$, eqn (45), imposed to the RVE and complemented by the higher-order boundary condition in terms of displacement normal derivative taken equal[8] to $D\overline{\boldsymbol{u}}^*$

$$\begin{cases} \overline{\boldsymbol{u}}^{SGE} = \overline{\boldsymbol{u}}^*, \\ \overline{D\boldsymbol{u}}^{SGE} = D\overline{\boldsymbol{u}}^*, \end{cases} \qquad \text{on } \partial\Omega^{SGE}_{eq}, \tag{49}$$

where $D\overline{\boldsymbol{u}}^*$ is the normal derivative of the displacement field (44).

According to the result presented in **Lemma 2** (Appendix A.2), the $\boldsymbol{\alpha}-\boldsymbol{\beta}$ energy term is null and the strain energy in $\Omega^{SGE}_{eq}$ is

$$\mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}\right), \tag{50}$$

where $D\overline{\boldsymbol{u}}^\alpha$ and $D\overline{\boldsymbol{u}}^{\beta\diamond *}$ are the contributions of the imposed normal derivative depending on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ terms in $D\overline{\boldsymbol{u}}^*$, respectively.

v) The energy minimization procedure, eqn (28), can be performed using the energy stored in the heterogeneous Cauchy material $\mathcal{W}^C_{RVE}$, eqn (48), and in the homogeneous SGE material $\mathcal{W}^{SGE}_{eq}$, eqn (50), so that the energy mismatch is given by

$$\mathcal{G}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = \mathcal{G}^\alpha\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) + \mathcal{G}^{\beta\diamond *}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) \tag{51}$$

where

$$\begin{aligned} \mathcal{G}^\alpha\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) &= \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^\alpha\right) - \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha\right), \\ \mathcal{G}^{\beta\diamond *}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) &= \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^{\beta\diamond *}\right) - \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}\right). \end{aligned} \tag{52}$$

[7] Considering that the RVE satisfies geometrical symmetry conditions, in addition to the geometrical properties **GP1** and **GP2**, it can be proven that the mutual energy is identically null even in the case of non-dilute suspension of inclusion

$$\mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^*\right) = \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^\alpha\right) + \mathcal{W}^C_{RVE}\left(\overline{\boldsymbol{u}}^{\beta\diamond *}\right), \qquad \forall f. \tag{47}$$

[8] The displacement field eqn (44) is the solution for a homogeneous SGE when boundary conditions (49) are imposed. It can be easily proven that the result of the proposed homogenization procedure holds when the higher-order boundary condition changes as $\overline{D\boldsymbol{u}}^{SGE} = D\overline{\boldsymbol{u}}^{RVE}$ since the strain energy developed in the SGE material is the same at the first order

$$\mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^{RVE}\right) = \mathcal{W}^{SGE}_{eq}\left(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*\right) + o(f).$$

Since only the local contribution (depending on $\mathbf{C}^{eq}$) arises in the SGE strain energy when the linear boundary displacement condition ($\boldsymbol{\beta}^{\diamond *} = \mathbf{0}$ and $\overline{\boldsymbol{u}}^{SGE} = \overline{\boldsymbol{u}}^{\alpha}$, $\overline{D\boldsymbol{u}}^{SGE} = D\overline{\boldsymbol{u}}^{\alpha}$) is imposed (while the non-local contribution depending on $\mathbf{A}^{eq}$ is identically null because higher-order stress and curvature are null), the energy mismatch $\mathcal{G}^{\alpha}$ due to the $\alpha$ terms is null by definition of $\mathbf{C}^{eq}$ (which is known from the first-order homogenization procedure)

$$\mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = \quad \mathcal{G}^{\alpha}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}\right) = 0. \tag{53}$$

Therefore, the proposed energy minimization procedure, based on linear and quadratic displacement boundary condition and leading to the definition of $\mathbf{A}^{eq}$, can be performed referring only to the $\beta^{\diamond *}$ terms,

$$\mathcal{G}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right) = \mathcal{G}^{\beta^{\diamond *}}\left(\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{eq}, \mathbf{A}^{eq}\right). \tag{54}$$

vi) Keeping into account the results presented in **Lemma 3** (Appendix A.3) and **Lemma 4** (Appendix A.4), the energy mismatch (54) is given by the difference of the following two terms

$$\mathcal{W}^{C}_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) = 2\rho^2 \Omega \mathbf{C}^{(1)}_{ijhk} \beta^{\diamond *}_{ijl} \beta^{\diamond *}_{hkl} + o(f). \tag{55}$$

and

$$\mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) = 2\Omega\left(\rho^2 \mathbf{C}^{eq}_{ijhk}\delta_{lm} + \mathbf{A}^{eq}_{jlikmh}\right)\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm} + o(f). \tag{56}$$

vii) Therefore, from eqns (36), (55) and (56), the annihilation of the strain energy gap $\mathcal{G}$, eqn (54) (between the real heterogeneous Cauchy and the equivalent homogeneous SGE materials) is represented by the condition

$$\left(f\rho^2 \tilde{\mathbf{C}}_{ijhk}\delta_{lm} + \mathbf{A}^{eq}_{jlikmh}\right)\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm} + o(f) = 0. \tag{57}$$

viii) The energy annihilation (57) has been obtained for a nonlinear displacement field $\boldsymbol{\beta}^{\diamond *}$, in equilibrium within a homogeneous material with local constitutive tensor $\mathbf{C}^{*}$. But, according to eqn (43), tensor $\mathbf{C}^{*}$ defines an arbitrary material, so that using this arbitrariness we obtain

$$\left(f\rho^2 \tilde{\mathbf{C}}_{ijhk}\delta_{lm} + \mathbf{A}^{eq}_{jlikmh}\right)\beta_{ijl}\beta_{hkm} + o(f) = 0, \tag{58}$$

where the components of $\boldsymbol{\beta}$ are unrestricted, except for the symmetry $\beta_{ijk}=\beta_{ikj}$. Eventually, the annihilation of energy mismatch $\mathcal{G}$, eqn (58), defines the non-local constitutive tensor $\mathbf{A}^{eq}$ for the equivalent SGE material as in eqn (35). □

## 5 Conclusions

Micro- or nano-structures embedded in solids introduce internal length-scales and nonlocal effects within the mechanical modelling, leading to higher-order theories. We have provided an analytical approach to the determination of the parameters defining an elastic higher-order (Mindlin) material, as the homogenization of a heterogeneous Cauchy elastic material, eqn (35). This result, obtained through the proposed homogenization procedure, is limited to the dilute approximation, but is not restricted to isotropy of the constituents and leaves a certain freedom to the shape of the inclusions. A perfect match between the elastic energies of the heterogeneous and homogeneous materials is obtained. Examples and results on material symmetry and positive definiteness are deferred to part II of this article (Bacca et al., 2013).

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# A Proofs of lemmas 1-4

## A.1 Lemma 1: Null mutual $\alpha$–$\beta$ energy term for the RVE at the first-order in concentration $f$

**Statement.** When a quadratic displacement $\overline{\boldsymbol{u}}^*$, eqn (45), is applied on the boundary of a RVE satisfying the geometrical properties **GP1** and **GP3**, the strain energy at first-order in $f$ is given by eqn (48).

**Proof.** By the superposition principle, the fields originated by the application of $\overline{\boldsymbol{u}}^* = \overline{\boldsymbol{u}}^\alpha + \overline{\boldsymbol{u}}^{\beta\diamond *}$ are given by the sum of the respective fields originated from the boundary conditions $\overline{\boldsymbol{u}}^\alpha$ and $\overline{\boldsymbol{u}}^{\beta\diamond *}$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\varepsilon}^\alpha(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\beta\diamond *}(\boldsymbol{x}), \qquad \boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{\sigma}^\alpha(\boldsymbol{x}) + \boldsymbol{\sigma}^{\beta\diamond *}(\boldsymbol{x}), \tag{A.1}$$

(the latter calculated through the constitutive eqn $(10)_1$) so that the strain energy $(26)_1$ becomes

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^*) = \mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha) + \mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta\diamond *}) + \underbrace{\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *})}_{mutual\ energy} \tag{A.2}$$

where

$$\begin{aligned}
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^\alpha_{ij}(\boldsymbol{x})\mathsf{C}_{ijhk}(\boldsymbol{x})\varepsilon^\alpha_{hk}(\boldsymbol{x}),\\
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta\diamond *}) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^{\beta\diamond *}_{ij}(\boldsymbol{x})\mathsf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{\beta\diamond *}_{hk}(\boldsymbol{x}),\\
\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}) &= \int_{\Omega_R} \varepsilon^\alpha_{ij}(\boldsymbol{x})\mathsf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{\beta\diamond *}_{hk}(\boldsymbol{x}).
\end{aligned} \tag{A.3}$$

Through two applications of the principle of virtual work[9] the mutual energy $(A.3)_3$ can be computed as

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}) = \alpha_{ij}\int_{\Omega_R} \sigma^{\beta\diamond *}_{ij}(\boldsymbol{x}), \tag{A.5}$$

[9]In the first application, the fields corresponding to the solution (A.1) are considered

$$\int_{\Omega_R} \varepsilon^\alpha_{ij}(\boldsymbol{x})\sigma^{\beta\diamond *}_{ij}(\boldsymbol{x}) = \int_{\partial\Omega_R} \overline{u}^\alpha_i(\boldsymbol{x}) t^{\beta\diamond *}_i(\boldsymbol{x}), \tag{A.4}$$

while in the second application, the kinematical field generated by the admissible displacement $\boldsymbol{u}^\alpha$ (44) within the RVE is considered so that the mutual energy (A.5) is obtained.

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which, using the constitutive relation $(10)_1$ and the symmetries of the local constitutive tensors $\mathbf{C}^{(1)}$ and $\mathbf{C}^{(2)}$, can be decomposed as the sum of two contributions

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^\alpha;\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) = \alpha_{ij}\mathbf{C}^{(1)}_{ijhk}\int_{\Omega_R} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) + \alpha_{ij}\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\int_{\Omega_{R_2}} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}). \tag{A.6}$$

Through two further applications of the divergence theorem and using the geometrical property **GP1** for the RVE,[10] the first term on the right-hand-side of eqn (A.6) results to be null

$$\alpha_{ij}\mathbf{C}^{(1)}_{ijhk}\int_{\Omega_R} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) = 0. \tag{A.9}$$

Introducing the mean value over a domain $\Omega$ of the function $f(\boldsymbol{x})$ as

$$\langle f(\boldsymbol{x})\rangle|_\Omega = \frac{1}{\Omega}\int_\Omega f(\boldsymbol{x}), \tag{A.10}$$

the second term on the right-hand-side of eqn (A.6) can be rewritten as

$$\alpha_{ij}\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\Omega_{R_2}\left\langle u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x})\right\rangle\Big|_{\Omega_{R_2}}. \tag{A.11}$$

Assuming the geometrical property **GP3** for the RVE, the displacement field in the presence of the inclusion is given by the asymptotic expansion in the volume fraction $f$

$$u^{\beta^{\diamond *}}_i = \beta^{\diamond *}_{ijk}x_j x_k + f^q\,\tilde{u}^{\beta^{\diamond *}}_i + o(f), \tag{A.12}$$

subject to the constraint

$$0 < q \leq 1, \tag{A.13}$$

and considering the geometrical property **GP1** for the RVE, together with the definition of volume fraction $f$, eqn (21), expression (A.11) becomes

$$f^{q+1}\Omega\,\alpha_{ij}\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\left\langle \tilde{u}^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x})\right\rangle\Big|_{\Omega_{R_2}}, \tag{A.14}$$

from which, considering the restriction on the power $q$ (A.13), the second term on the right-hand-side of eqn (A.6) is null at first-order in $f$

$$\alpha_{ij}\left(\mathbf{C}^{(2)}_{ijhk} - \mathbf{C}^{(1)}_{ijhk}\right)\int_{\Omega_{R_2}} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) = o(f). \tag{A.15}$$

Considering results (A.9) and (A.15), the mutual energy in the RVE $(A.3)_3$ is null at first-order in $f$ and proposition (48) follows. □

[10] In the first application of the divergence theorem, $u^{\beta^{\diamond *}} = \overline{u}^{\beta^{\diamond *}}$, eqn (45), is considered on the boundary $\partial\Omega_R$, so that

$$\int_{\Omega_R} u^{\beta^{\diamond *}}_{h,k}(\boldsymbol{x}) = \beta^{\diamond *}_{hlm}\int_{\partial\Omega_R} n_k x_l x_m, \tag{A.7}$$

while, in the second application, the kinematically admissible displacement field $\overline{u}^{\beta^{\diamond *}}$, eqn (44), is considered within the RVE, yielding

$$\beta^{\diamond *}_{hlm}\int_{\partial\Omega_R} n_k x_l x_m = 2\beta^{\diamond *}_{hlk}\int_{\Omega_R} x_l, \tag{A.8}$$

so that the geometrical property **GP1** for the RVE leads to eqn (A.9).

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### A.2 Lemma 2: Null mutual $\boldsymbol{\alpha}$–$\boldsymbol{\beta}$ energy term for the homogeneous SGE

**Statement.** When a quadratic displacement $\overline{\boldsymbol{u}}^*$, eqn (45), and the normal component of its derivative $D\overline{\boldsymbol{u}}^*$ are applied on the boundary of a SGE satisfying the geometrical property **GP1**, the strain energy is given by eqn (50).

**Proof.** By the superposition principle, the fields originated by the application of the boundary conditions ($\overline{\boldsymbol{u}}^* = \overline{\boldsymbol{u}}^\alpha + \overline{\boldsymbol{u}}^{\beta\diamond *}$, $D\overline{\boldsymbol{u}}^* = D\overline{\boldsymbol{u}}^\alpha + D\overline{\boldsymbol{u}}^{\beta\diamond *}$) can be obtained as the sum of the respective fields arising from the boundary conditions ($\overline{\boldsymbol{u}}^\alpha$, $D\overline{\boldsymbol{u}}^\alpha$) and ($\overline{\boldsymbol{u}}^{\beta\diamond *}$, $D\overline{\boldsymbol{u}}^{\beta\diamond *}$) in the forms

$$\begin{aligned} \boldsymbol{\varepsilon}(\boldsymbol{x}) &= \boldsymbol{\varepsilon}^\alpha(\boldsymbol{x}) + \boldsymbol{\varepsilon}^{\beta\diamond *}(\boldsymbol{x}), \qquad & \boldsymbol{\chi}(\boldsymbol{x}) &= \boldsymbol{\chi}^\alpha(\boldsymbol{x}) + \boldsymbol{\chi}^{\beta\diamond *}(\boldsymbol{x}), \\ \boldsymbol{\sigma}(\boldsymbol{x}) &= \boldsymbol{\sigma}^\alpha(\boldsymbol{x}) + \boldsymbol{\sigma}^{\beta\diamond *}(\boldsymbol{x}), \qquad & \boldsymbol{\tau}(\boldsymbol{x}) &= \boldsymbol{\tau}^\alpha(\boldsymbol{x}) + \boldsymbol{\tau}^{\beta\diamond *}(\boldsymbol{x}), \end{aligned} \tag{A.16}$$

(the latter calculated through the constitutive eqn (10)) so that the strain energy $(26)_2$ becomes

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^*, D\overline{\boldsymbol{u}}^*) = \underbrace{\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha) + \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *})}_{direct\ energy} + \underbrace{\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *})}_{mutual\ energy} \tag{A.17}$$

where

$$\begin{aligned} \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha) &= \frac{1}{2}\int_{\Omega_{eq}} \left[\varepsilon_{ij}^\alpha(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^\alpha(\boldsymbol{x}) + \chi_{ijl}^\alpha(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^\alpha(\boldsymbol{x})\right], \\ \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) &= \frac{1}{2}\int_{\Omega_{eq}} \left[\varepsilon_{ij}^{\beta\diamond *}(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^{\beta\diamond *}(\boldsymbol{x}) + \chi_{ijl}^{\beta\diamond *}(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^{\beta\diamond *}(\boldsymbol{x})\right], \\ \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) &= \int_{\Omega_{eq}} \left[\varepsilon_{ij}^\alpha(\boldsymbol{x})\mathsf{C}_{ijhk}^{eq}\varepsilon_{hk}^{\beta\diamond *}(\boldsymbol{x}) + \chi_{ijl}^\alpha(\boldsymbol{x})\mathsf{A}_{ijlhkm}^{eq}\chi_{hkm}^{\beta\diamond *}(\boldsymbol{x})\right]. \end{aligned} \tag{A.18}$$

Application of the boundary condition ($\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha$) on $\partial\Omega_{eq}$ leads to the displacement field $\boldsymbol{u}^\alpha(\boldsymbol{x})$, eqn (44), so that $\boldsymbol{\chi}^\alpha(\boldsymbol{x}) = \mathbf{0}$ and, considering the symmetries of the equivalent local constitutive tensor $\mathsf{C}^{eq}$, the mutual energy simplifies in the local contribution

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^\alpha, D\overline{\boldsymbol{u}}^\alpha; \overline{\boldsymbol{u}}^{\beta\diamond *}, D\overline{\boldsymbol{u}}^{\beta\diamond *}) = \alpha_{ij}\mathsf{C}_{ijhk}^{eq}\int_{\Omega_{eq}} u_{h,k}^{\beta\diamond *}(\boldsymbol{x}). \tag{A.19}$$

Through two applications of the divergence theorem and using the geometrical property **GP1** of the SGE, the mutual energy (A.19) is null and then proposition (50) follows. □

### A.3 Lemma 3: $\boldsymbol{\beta}$ term in the strain energy $\mathcal{W}_{RVE}^{C}$ at first-order in $f$

**Statement.** When a quadratic displacement $\overline{\boldsymbol{u}}^{\beta\diamond *}$, eqn (45) with $\boldsymbol{\alpha} = \mathbf{0}$, is applied on the RVE boundary, the strain energy at first-order in the concentration $f$ is given by eqn (55).

**Proof.** The strain energy $\mathcal{W}_{RVE}^{C}(\overline{\boldsymbol{u}}^{\beta\diamond *})$ stored in the RVE, when a quadratic displacement field $\overline{\boldsymbol{u}}^{\beta\diamond *}$ (45) is applied on its boundary $\partial\Omega_{RVE}$, is bounded by (see Gurtin, 1972)

$$\int_{\partial\Omega_{RVE}} \sigma_{ij}^{SA} n_i \overline{u}_j^{\beta\diamond *} - \mathcal{U}_{RVE}^{C}(\boldsymbol{\sigma}^{SA}) \leq \mathcal{W}_{RVE}^{C}(\overline{\boldsymbol{u}}^{\beta\diamond *}) \leq \mathcal{W}_{RVE}^{C}(\boldsymbol{\varepsilon}^{KA}), \tag{A.20}$$

where $\boldsymbol{\varepsilon}^{KA}$ is a kinematically admissible (satisfying the kinematic compatibility relation $(1)_1$ and the imposed displacement boundary conditions) strain field, $\boldsymbol{\sigma}^{SA}$ is a statically admissible (satisfying the equilibrium condition, eqn (4) with $\boldsymbol{\tau} = \mathbf{0}$) stress field, while $\mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA})$ and $\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA})$ are respectively the following stress and strain energies

$$\begin{aligned} \mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA}) &= \frac{1}{2}\int_{\Omega_R} \sigma^{SA}_{ij}(\boldsymbol{x})\mathsf{C}^{-1}_{ijhk}(\boldsymbol{x})\sigma^{SA}_{hk}(\boldsymbol{x}), \\ \mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA}) &= \frac{1}{2}\int_{\Omega_R} \varepsilon^{KA}_{ij}(\boldsymbol{x})\mathsf{C}_{ijhk}(\boldsymbol{x})\varepsilon^{KA}_{hk}(\boldsymbol{x}). \end{aligned} \tag{A.21}$$

Considering the kinematically admissible strain field

$$\varepsilon^{KA}_{ij} = (\beta^{\diamond*}_{ijk} + \beta^{\diamond*}_{jik})x_k, \tag{A.22}$$

and assuming the geometrical properties **GP2** and **GP3**, an estimate for the *upper bound* in eqn (A.20) is the strain energy $\mathcal{W}^C_{RVE}(\boldsymbol{\varepsilon}^{KA})$ given by eqn (B.5)$_1$ (Appendix B.1), so that

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond*}}) \leq 2\rho^2\Omega\mathsf{C}^{(1)}_{ijhk}\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkl} + o(f). \tag{A.23}$$

Considering now the statically admissible stress field

$$\sigma^{SA}_{ij} = 2\mathsf{C}^{*}_{ijhk}\beta^{\diamond*}_{hkl}x_l, \tag{A.24}$$

where $\mathsf{C}^*$ is a first-order perturbation in $f$ to the material matrix $\mathsf{C}^{(1)}$, eqn (43), and assuming the geometrical property **GP2**, the stress energy $\mathcal{U}^C_{RVE}(\boldsymbol{\sigma}^{SA})$ is given by eqn (B.5)$_2$ (Appendix B.1). Moreover, since the application of the divergence theorem yields

$$\int_{\partial\Omega_R} \sigma^{SA}_{ij}n_i\overline{u}^{\beta^{\diamond*}}_j = 4\rho^2\Omega\left(\mathsf{C}^{(1)}_{ijhk} + f\hat{\mathsf{C}}_{ijhk}\right)\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkl}, \tag{A.25}$$

an estimate is obtained for the *lower bound* in eqn (A.20) as

$$\mathcal{W}^C_{RVE}(\overline{\boldsymbol{u}}^{\beta^{\diamond*}}) \geq 2\rho^2\Omega\mathsf{C}^{(1)}_{ijhk}\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkl} + o(f), \tag{A.26}$$

which, together with the upper bound (A.23), leads to eqn (55). □

### A.4 Lemma 4: $\boldsymbol{\beta}$ term in the strain energy $\mathcal{W}^{SGE}_{eq}$ at first-order in $f$.

**Statement.** When a quadratic displacement $\overline{\boldsymbol{u}}^{\beta^{\diamond*}}$, eqn (45) with $\boldsymbol{\alpha} = \mathbf{0}$, and the normal component of its gradient $D\overline{\boldsymbol{u}}^{\beta^{\diamond*}}$ are imposed on the boundary of the homogeneous SGE equivalent material, the strain energy at first-order in the concentration $f$ is given by eqn (56).

**Proof.** The strain energy $\mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}}^{\beta^{\diamond*}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond*}})$ stored in the SGE, when a quadratic displacement field $\overline{\boldsymbol{u}}^{\beta^{\diamond*}}$ (45) and the normal component of its gradient $D\overline{\boldsymbol{u}}^{\beta^{\diamond*}}$ are imposed on its boundary $\partial\Omega_{eq}$, is bounded as (Appendix C)

$$\begin{aligned} &\int_{\partial\Omega_{eq}}\left(t^{SA}_i\overline{u}^{\beta^{\diamond*}}_i + T^{SA}_i D\overline{u}^{\beta^{\diamond*}}_i\right) + \int_{\Gamma_{eq}}\Theta^{SA}_i\overline{u}^{\beta^{\diamond*}}_i - \mathcal{U}^{SGE}_{eq}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) \leq \\ &\leq \mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}}^{\beta^{\diamond*}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond*}}) \leq \mathcal{W}^{SGE}_{eq}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA}), \end{aligned} \tag{A.27}$$

with

$$\begin{cases} t^{SA}_k = n_j\sigma^{SA}_{jk} - n_in_jD\tau^{SA}_{ijk} - 2n_jD_i\tau^{SA}_{ijk} + (n_in_jD_ln_l - D_jn_i)\,\tau^{SA}_{ijk}, \\ T^{SA}_k = n_in_j\tau^{SA}_{ijk}, \end{cases} \qquad \text{on } \partial\Omega_{eq}, \tag{A.28}$$

and

$$\Theta_k^{SA} = [[e_{mlj} n_i s_m n_l \tau_{ijk}^{SA}]], \qquad \text{on } \Gamma_{eq}, \tag{A.29}$$

where $\boldsymbol{\varepsilon}^{KA}$ and $\boldsymbol{\chi}^{KA}$ are kinematically admissible strain and curvature fields (satisfying the kinematic compatibility relation (1) and the imposed displacement boundary conditions), $\boldsymbol{\sigma}^{SA}$ and $\boldsymbol{\tau}^{SA}$ are statically admissible stress and double-stress fields (satisfying the equilibrium equation (4)), while $\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA})$ and $\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA})$ are respectively the stress and the strain energies given by

$$\begin{aligned}
\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) &= \frac{1}{2}\int_{\Omega_{eq}} \sigma_{ij}^{SA}(\boldsymbol{x}) \mathbf{C}_{ijhk}^{eq^{-1}} \sigma_{hk}^{SA}(\boldsymbol{x}) + \frac{1}{2}\int_{\Omega_{eq}} \tau_{ijh}^{SA}(\boldsymbol{x}) \mathbf{A}_{ijhklm}^{eq^{-1}} \tau_{klm}^{SA}(\boldsymbol{x}), \\
\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA}) &= \frac{1}{2}\int_{\Omega_{eq}} \varepsilon_{ij}^{KA}(\boldsymbol{x}) \mathbf{C}_{ijhk}^{eq} \varepsilon_{hk}^{KA}(\boldsymbol{x}) + \frac{1}{2}\int_{\Omega_{eq}} \chi_{ijh}^{KA}(\boldsymbol{x}) \mathbf{A}_{ijhklm}^{eq} \chi_{klm}^{KA}(\boldsymbol{x}).
\end{aligned} \tag{A.30}$$

Considering the kinematically admissible strain $\boldsymbol{\varepsilon}^{KA}$ (A.22) and curvature field

$$\chi_{ijk}^{KA} = 2\beta_{kij}^{\diamond *}, \tag{A.31}$$

and assuming the geometrical property **GP2**, an estimate for the *upper bound* in eqn (A.27) is the strain energy $\mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA})$ given by eqn (B.8)$_1$ (Appendix B.2) as

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \leq 2\Omega \beta_{ijl}^{\diamond *} \beta_{hkm}^{\diamond *} \left(\rho^2 \mathbf{C}_{ijhk}^{eq} \delta_{lm} + \mathbf{A}_{jlikmh}^{eq}\right). \tag{A.32}$$

Considering the statically admissible stress $\boldsymbol{\sigma}^{SA}$ (A.24) and double-stress field

$$\tau_{jli}^{SA} = 2\mathbf{A}_{jlikmh}^{eq} \beta_{hkm}^{\diamond *}, \tag{A.33}$$

where $\mathbf{C}^*$ is a first-order perturbation in $f$ to the material matrix $\mathbf{C}^{eq}$, eqn (42), and assuming the geometrical property **GP2**, the stress energy $\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA})$ is given by eqn (B.9) (Appendix B.2). Moreover, since the application of the divergence theorem yields

$$\int_{\partial\Omega_{eq}} \left(t_i^{SA} \overline{u}_i^{\beta^{\diamond *}} + T_i^{SA} D\overline{u}_i^{\beta^{\diamond *}}\right) + \int_{\Gamma_{eq}} \Theta_i^{SA} \overline{u}_i^{\beta^{\diamond *}} = 4\rho^2 \Omega \left[\mathbf{C}_{ijhk}^{eq} + f\left(\hat{\mathbf{C}}_{ijhk} - \tilde{\mathbf{C}}_{ijhk}\right)\right] \beta_{ijn}^{\diamond *} \beta_{hkn}^{\diamond *}, \tag{A.34}$$

an estimate is obtained for the *lower bound* in eqn (A.27) as

$$\mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}}) \geq 2\Omega \beta_{ijl}^{\diamond *} \beta_{hkm}^{\diamond *} \left(\rho^2 \mathbf{C}_{ijhk}^{eq} \delta_{lm} + \mathbf{A}_{jlikmh}^{eq}\right) + o(f), \tag{A.35}$$

which, together with the upper bound (A.32), leads to eqn (56). □

# B Elastic energies based on the kinematically admissible displacement field $\boldsymbol{u}^{\beta^{\diamond *}}$ (44)

In this Appendix it is assumed $\boldsymbol{\alpha} = \mathbf{0}$. The field $\boldsymbol{u}^{\beta^{\diamond *}}$, eqn (44), is a kinematically admissible displacement for both boundary conditions $\overline{\boldsymbol{u}}^{\beta^{\diamond *}}$, eqn (46), and $(\overline{\boldsymbol{u}}^{\beta^{\diamond *}}, D\overline{\boldsymbol{u}}^{\beta^{\diamond *}})$, eqn (49), applied on the boundary of the RVE and the SGE, respectively. The related strain and stress energies in the RVE and in the SGE are obtained below.

- In Section B.1 the strain energies are computed with the kinematically admissible deformation $\boldsymbol{\varepsilon}^{KA}$, eqn (A.22), and curvature $\boldsymbol{\chi}^{KA}$, eqn (A.31), originated by the kinematically admissible displacement $\boldsymbol{u}^{\beta^{\diamond *}}$, eqn (44);

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- In Section B.2 the stress energies are computed with the statically admissible stress $\boldsymbol{\sigma}^{SA}$, eqn (A.24), and double-stress $\boldsymbol{\tau}^{SA}$, eqn (A.33), originated by the above mentioned kinematically admissible fields $\boldsymbol{\varepsilon}^{KA}$ and $\boldsymbol{\chi}^{KA}$ within a homogeneous material with constitutive tensors $\mathbf{C}^*$ and $\mathbf{A}^{eq}$.

## B.1 Strain and stress energies in the RVE

The kinematically admissible deformation $\boldsymbol{\varepsilon}^{KA}$, eqn (A.22), and the statically admissible stress $\boldsymbol{\sigma}^{SA}$, eqn (A.24), provide the strain and stress energies (A.21) in the RVE

$$\begin{aligned} \mathcal{W}^{C}_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \int_{\Omega} 2\mathbf{C}_{ijhk}(\boldsymbol{x})\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm}x_l x_m, \\ \mathcal{U}^{C}_{RVE}(\boldsymbol{\sigma}^{SA}) = & \int_{\Omega} 2\mathbf{C}^*_{ijlm}\mathbf{C}^{-1}_{ijhk}(\boldsymbol{x})\mathbf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rst}x_n x_t, \end{aligned} \tag{B.1}$$

which, introducing the definition (33) of the Euler tensor of inertia $\boldsymbol{E}$, can be rewritten as

$$\begin{aligned} \mathcal{W}^{C}_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\left[\mathbf{C}^{(1)}_{ijhk}E_{lm}(\Omega^C_1) + \mathbf{C}^{(2)}_{ijhk}E_{lm}(\Omega^C_2)\right]\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm}, \\ \mathcal{U}^{C}_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\mathbf{C}^*_{ijlm}\left\{\mathbf{C}^{(1)^{-1}}_{ijhk}E_{nt}(\Omega^C_1) + \mathbf{C}^{(2)^{-1}}_{ijhk}E_{nt}(\Omega^C_2)\right\}\mathbf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rst}. \end{aligned} \tag{B.2}$$

Assuming the geometrical property **GP2** and considering the identity (30), the strain and stress energies (B.2) simplify as

$$\begin{aligned} \mathcal{W}^{C}_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\rho^2\Omega\left\{\mathbf{C}^{(1)}_{ijhk} - f\left(\frac{\rho^{(2)}}{\rho}\right)^2\left[\mathbf{C}^{(1)}_{ijhk} - \mathbf{C}^{(2)}_{ijhk}\right]\right\}\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl}, \\ \mathcal{U}^{C}_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\rho^2\Omega\mathbf{C}^*_{ijlm}\left\{\mathbf{C}^{(1)^{-1}}_{ijhk} - f\left(\frac{\rho^{(2)}}{\rho}\right)^2\left[\mathbf{C}^{(2)^{-1}}_{ijhk} - \mathbf{C}^{(1)^{-1}}_{ijhk}\right]\right\}\mathbf{C}^*_{hkrs}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rsn}. \end{aligned} \tag{B.3}$$

Assuming the geometrical property **GP3**

$$\rho^{(2)} = \tilde{\rho}^{(2)}f^r + o(f), \tag{B.4}$$

with $0 < r \leq 1$, and $\mathbf{C}^*$ as a first-order perturbation in $f$ to the material matrix $\mathbf{C}^{(1)}$, eqn (43), the strain and the stress energies are given in the dilute case ($f \ll 1$) by

$$\begin{aligned} \mathcal{W}^{C}_{RVE}(\boldsymbol{\varepsilon}^{KA}) = & \; 2\rho^2\Omega\mathbf{C}^{(1)}_{ijhk}\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl} + o(f), \\ \mathcal{U}^{C}_{RVE}(\boldsymbol{\sigma}^{SA}) = & \; 2\rho^2\Omega\left(\mathbf{C}^{(1)}_{ijhk} + 2f\hat{\mathbf{C}}_{ijhk}\right)\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkl} + o(f). \end{aligned} \tag{B.5}$$

## B.2 Strain and stress energies in the SGE

The kinematically admissible deformation and curvature fields [$\boldsymbol{\varepsilon}^{KA}$, eqn (A.22); $\boldsymbol{\chi}^{KA}$, eqn (A.31)] together with the statically admissible stress and double-stress fields [$\boldsymbol{\sigma}^{SA}$, eqn (A.24); $\boldsymbol{\tau}^{SA}$, eqn (A.33)] provide the strain and stress energies (A.30) in the SGE

$$\begin{aligned} \mathcal{W}^{SGE}_{eq}(\boldsymbol{\varepsilon}^{KA}, \boldsymbol{\chi}^{KA}) = & \int_{\Omega} 2\left[\mathbf{C}^{eq}_{ijhk}x_l x_m + \mathbf{A}^{eq}_{jlikmh}\right]\beta^{\diamond *}_{ijl}\beta^{\diamond *}_{hkm}, \\ \mathcal{U}^{SGE}_{eq}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) = & \int_{\Omega} 2\left\{\mathbf{C}^*_{ijlm}\mathbf{C}^{eq^{-1}}_{ijhk}\mathbf{C}^*_{hkrs}x_n x_t + \mathbf{A}^{eq}_{mnlstr}\right\}\beta^{\diamond *}_{lmn}\beta^{\diamond *}_{rst}, \end{aligned} \tag{B.6}$$

which, introducing the definition (33) for the Euler tensor of inertia $\boldsymbol{E}$, can be rewritten as

$$\begin{aligned} \mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) = &\ 2\left[\mathbf{C}^{eq}_{ijhk}E_{lm}(\Omega^{SGE}_{eq})+\Omega^{SGE}_{eq}\mathbf{A}^{eq}_{jlikmh}\right]\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkm}, \\ \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) = &\ 2\left\{\mathbf{C}^{*}_{ijlm}\mathbf{C}^{eq^{-1}}_{ijhk}\mathbf{C}^{*}_{hkrs}E_{nt}(\Omega^{SGE}_{eq})+\Omega^{SGE}_{eq}\mathbf{A}^{eq}_{mnlstr}\right\}\beta^{\diamond*}_{lmn}\beta^{\diamond*}_{rst}. \end{aligned} \tag{B.7}$$

Assuming the geometrical property **GP2**, the strain and stress energies (B.7) simplify as

$$\begin{aligned} \mathcal{W}_{eq}^{SGE}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) = &\ 2\Omega\left[\rho^2\mathbf{C}^{eq}_{ijhk}\delta_{lm}+\mathbf{A}^{eq}_{jlikmh}\right]\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkm}, \\ \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) = &\ 2\Omega\left\{\rho^2\mathbf{C}^{*}_{ijlm}\mathbf{C}^{eq^{-1}}_{ijhk}\mathbf{C}^{*}_{hkrs}\delta_{nt}+\mathbf{A}^{eq}_{mnlstr}\right\}\beta^{\diamond*}_{lmn}\beta^{\diamond*}_{rst}. \end{aligned} \tag{B.8}$$

Finally, assuming $\mathbf{C}^*$ as a first-order perturbation in $f$ to the equivalent local tensor $\mathbf{C}^{eq}$, eqn (42), the stress energy is given in the dilute case ($f \ll 1$) by

$$\mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA}) = 2\Omega\left\{\rho^2\left[\mathbf{C}^{eq}_{ijhk}+2f\left(\hat{\mathbf{C}}_{ijhk}-\tilde{\mathbf{C}}_{ijhk}\right)\right]\delta_{lm}+\mathbf{A}^{eq}_{jlikmh}\right\}\beta^{\diamond*}_{ijl}\beta^{\diamond*}_{hkm}+o(f). \tag{B.9}$$

# C Energy bounds for SGE Material

**Statement.** When boundary displacement conditions $\overline{\boldsymbol{u}}$, $\overline{D\boldsymbol{u}}$ are imposed on the boundary $\partial\Omega_{eq}$ of a SGE, the strain energy $\mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}})$ is bounded as

$$\int_{\partial\Omega_{eq}}\left(t_i^{SA}\overline{u}_i+T_i^{SA}\overline{Du}_i\right)+\int_{\Gamma_{eq}}\Theta_i^{SA}\overline{u}_i-\mathcal{U}^{SGE}_{eq}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA})\leq\mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}})\leq\mathcal{W}^{SGE}_{eq}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}), \tag{C.1}$$

where $\boldsymbol{\varepsilon}^{KA}$ and $\boldsymbol{\chi}^{KA}$ are kinematically admissible strain and curvature fields (satisfying the kinematic compatibility relation (1) and the imposed displacement boundary conditions), $\boldsymbol{\sigma}^{SA}$ and $\boldsymbol{\tau}^{SA}$ are statically admissible stress and double-stress fields (satisfying the equilibrium equation (4)) and the other statically admissible quantities $\boldsymbol{t}^{SA}$, $\boldsymbol{T}^{SA}$ and $\boldsymbol{\Theta}^{SA}$ are given by eqns (A.28) and (A.29), while $\mathcal{U}^{SGE}_{eq}(\boldsymbol{\sigma}^{SA},\boldsymbol{\tau}^{SA})$ and $\mathcal{W}^{SGE}_{eq}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA})$ are respectively the stress and the strain energies, eqns (A.30)$_1$ and (A.30)$_2$.

**Proof.** Considering the displacement field $\boldsymbol{u}^{eq}$ solution to the displacement boundary conditions $\overline{\boldsymbol{u}}$, $\overline{D\boldsymbol{u}}$ and the related statical fields $\boldsymbol{\sigma}^{eq}$ and $\boldsymbol{\tau}^{eq}$ in equilibrium, through the difference fields $\Delta\boldsymbol{\varepsilon}^{KA}$, $\Delta\boldsymbol{\chi}^{KA}$, $\Delta\boldsymbol{\sigma}^{SA}$, $\Delta\boldsymbol{\tau}^{SA}$ the kinematically and statically admissible fields can be defined as

$$\begin{aligned} \boldsymbol{\varepsilon}^{KA} &= \boldsymbol{\varepsilon}^{eq}+\Delta\boldsymbol{\varepsilon}^{KA}, \qquad \boldsymbol{\chi}^{KA}=\boldsymbol{\chi}^{eq}+\Delta\boldsymbol{\chi}^{KA}, \\ \boldsymbol{\sigma}^{SA} &= \boldsymbol{\sigma}^{eq}+\Delta\boldsymbol{\sigma}^{SA}, \qquad \boldsymbol{\tau}^{SA}=\boldsymbol{\tau}^{eq}+\Delta\boldsymbol{\tau}^{SA}. \end{aligned} \tag{C.2}$$

Using the discrepancy fields $\Delta\boldsymbol{\varepsilon}^{KA}$ and $\Delta\boldsymbol{\chi}^{KA}$ the term representing the *upper bound* in eqn (C.1) can be rewritten as

$$\begin{aligned} \mathcal{W}^{SGE}_{eq}(\boldsymbol{\varepsilon}^{KA},\boldsymbol{\chi}^{KA}) = &\ \mathcal{W}^{SGE}_{eq}(\overline{\boldsymbol{u}},\overline{D\boldsymbol{u}})+\mathcal{W}^{SGE}_{eq}(\Delta\boldsymbol{\varepsilon}^{KA},\Delta\boldsymbol{\chi}^{KA}) \\ &+\int_{\Omega_{eq}}\left(\mathbf{C}_{ijhk}\varepsilon^{eq}_{ij}\Delta\varepsilon^{KA}_{hk}+\mathbf{A}_{ijklmn}\chi^{eq}_{ijk}\Delta\chi^{KA}_{lmn}\right), \end{aligned} \tag{C.3}$$

which provides a proof to the upper bound, since the strain energy is positive definite and the third term in the RHS of eqn (C.3) is null by the principle of virtual work (3) with $\Delta \boldsymbol{u} = \Delta D\boldsymbol{u} = \boldsymbol{0}$ on the boundary.

Using the discrepancy fields $\Delta\boldsymbol{\sigma}^{KA}$ and $\Delta\boldsymbol{\tau}^{KA}$ the term representing the *lower bound* in eqn (C.1) can be rewritten as

$$\int_{\partial\Omega_{eq}} (t_i^{SA}\overline{u}_i + T_i^{SA}\overline{Du}_i) + \int_{\Gamma_{eq}} \Theta_i^{SA}\overline{u}_i - \mathcal{U}_{eq}^{SGE}(\boldsymbol{\sigma}^{SA}, \boldsymbol{\tau}^{SA}) = \mathcal{W}_{eq}^{SGE}(\overline{\boldsymbol{u}}, \overline{D\boldsymbol{u}}) - \mathcal{U}_{eq}^{SGE}(\Delta\boldsymbol{\sigma}^{SA}, \Delta\boldsymbol{\tau}^{SA}) \tag{C.4}$$

which provides a proof to the lower bound, since the strain energy is positive definite. □

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